Baby Steps through Basic Algebra

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Contents

CHAPTER 1

The Basics

1.1. Preliminaries

"Our priority is not to find the fastest *or the* easiest *way to get the answer. We seek the path to the answer that will lead to the best and most profound understanding of the solution"*

Gameplan 1.1

(1) *purpose of this course*
(2) *the way we do business*
(3) *how to justify*

The Purpose of This Course

For many readers, this course may serve as a prerequisite or step towards completing a degree requirement. That being said, we consider the question: *What is the point of studying math?* First and foremost, its just for fun. Do not expect to apply this directly to everyday life. In real life you will never have to solve a riddle to determine how old someone is, or how long it will take you to meet your friend if she rides on a bike and you drive in a car from opposite sides of town. In real life, unless you are a rocket scientist, all you *need* to know is how to balance your checkbook and count the change you get when you buy your latte. Better, think of this as the PE class for your brain. If all goes well you should get a good few sets of push-ups for your brain every day. These may cause a little discomfort at times. The satisfaction, fun, and the challenge you will get will make it all worth it. If this is not convincing enough, well then, *ask not what math can do for you, but what can you do for math!*

What We Seek

We seek *understanding*. The solution to any problem is really insignificant. Understanding the process for getting there is all that matters. Our priority is not to find the *fastest* or the *easiest* way to get the answer, rather we seek the path to the answer that will lead to the best and most profound understanding of the solution. This must be emphasized, we put all our might and focus on *understanding* the process, never ever ever on the fastest or easiest way to the answer.

Ask Why

Part of understanding is asking and knowing *why* we do what we do. Humans are naturally curious. Children are the perfect math students because they are always asking 'why?' We will celebrate this innate curiosity and embrace the question 'why?' To us, asking 'why?' is the same as saying, "I want to have a better understanding of what I am doing, I don't believe everything I am told. I am a thinking, curious being, capable of thinking critically of what I am being told!" That is a beautiful attitude.

We Show and Justify Every Step

By showing every single step we reduce the chances of making a mistake. By justifying every step will understand *why* we do what we do, and we will only perform steps we *know* to be correct. For example, in solving an equation it may be necessary to 'cancel' a $2x$ from the equation. The most common problem is that students are not sure when 'cancelling' is a legitimate step. Requiring justification for every step insures that every step is unquestionably correct *and* unquestionably understood! ... and that is as good as it gets!

What is Justification?

For us, justification will be an *Axioms* we have accepted, a *Definition* we have accepted, or a *Theorem*. Axioms are truths we hold self evident while definitions are words we make up and accept in lieu of other words or phrases. For examples, we could agree that *all natural numbers that are multiples of two* will be called *even numbers*. This would be a perfect examples of a definition. All it says is that we may use *even numbers* instead of the entire phrase *all natural numbers that are multiples of two*. There is nothing special about the word *even*. We could just as easily have chosen *sweet numbers* to describe such numbers. An example of an axiom, is the following statement: *"adding a first integer plus a second integer gives the same result as adding the second integer plus the first integer"*. Definitions and Axioms are the building blocks. They are to a mathematician what paint tubes are to an artist. An artist will take the paint tubes, a brush and a canvas and make the most remarkable paintings. A mathematician will take the axioms and definitions a pencil and a blank piece of paper and make the most remarkable theorems. Some paintings describe the world while others are tributes to beauty, and elegance. Similarly, in math, some theorems describe the ways of the world while others are there just for the sheer beauty of them.

For the first part of the book every single step will be justified by either an *Axiom*, a *Definition*, or a proven *Theorem*. Some steps/justifications will be performed hundreds of times. After some point these become very well understood and we may elect to justify this steps by simply saying *by inspection*. These are the only acceptable justification: a definition, axiom, theorem, or by inspection. 'because my teacher said so', 'because the calculator says so', 'because the book said so' or any similar reasoning is not valid justification!!!

What you can expect

This method of doing math will definitely help build your brain muscles. The simple act of being curious and asking why can be one of the healthiest habits for your brain. You can expect to gain a better understanding of math than you ever imagined. Five out of ever four math teachers recommend this method of doing math. Stick with it, it will change your life!

This first chapter will require some patience and humbleness. Consider this chapter the planting stage. The harvest will begin promptly hereafter.

1.2. The Natural Numbers

"Thus, if a stranger stopped you in the middle of the street and asked you 'what is 5+3?', you could reply 'that is 8. Not only do i know it's 8, I can also tell you why it is 8.'... "

Gameplan 1.2

(1) *what are 'the naturals'*
(2) *what is the symbol for 'natural numbers'*
(3) *what is a 'binary operation'*
(4) *learn to add naturals*
(5) *learn to multiply naturals*

The Idea

The key to understanding algebra is understanding *addition, subtraction, multiplication*, and *division*. The fancy name for these creatures is *binary operations*. The deeper your understanding of these operations, the deeper your understanding of *algebra*. Consider the examples, $3+5$, $-5+2$, $\frac{2}{3}+\frac{-5}{7}$, $dog + cat$, or $cat + (3cats + \pi x^2)$. The first one is very easy, the others demand a more solid understanding of what we mean when we write $'+'$.

The idea here is to learn how to add, subtract, multiply and divide really, really, well. In order to learn these operations we will need a clean start. At this point, forget about everything you have ever learned about algebra. We will learn small bits at a time, but we will learn like we've never learned before.

First, we will learn what the *natural numbers* are.

The Natural Numbers

The natural numbers are the numbers most people learn when the are very young, *one, two, three,...*. Whenever we talk about the *natural numbers*, we are referring to precisely these numbers. They are also represented by the symbol, $\mathbb{N}$. So whenever you see the symbol,$\mathbb{N}$, you should be thinking about the numbers $1, 2, 3, 4, 5, ...$ This is also a good time to point out that the three dots after 5 mean the the list goes on. Rather than writing all the natural numbers (there are infinite many) we simply write the dots to indicate that the list continues. We could summarize the definition of *natural numbers by;*

$$\text{Natural Numbers} = \{1, 2, 3, 4, 5, \dots\}$$

We could also summarize the definition using the symbol for natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

From this point on, we will assume we all know what *natural numbers* are. We will also use the symbol $\mathbb{N}$ with no further explanation, under the understanding that whenever we see '$\mathbb{N}$' we think 'the numbers $\{1, 2, 3, 4, 5, \dots\}$'

There are many numbers that are not natural numbers. For example, 0 is not a natural number, nor are negative numbers, or most fractions. Actually the list of numbers that are not natural is immense and for now, we will be content with just knowing the natural numbers. As we progress we will meet other families of numbers.

Adding Natural Numbers

At this point we will assume we know how to add natural numbers. You may have seen 'adding tables' before. For example, to add 3+5, we look at column 3 and row 5 to obtain 8. It may seem silly now, be patient. The reason for doing this is so that we know exactly why we do what we do, and what we can do and can not do algebraically. We will take the addition table for natural numbers to be an axiom, something we accept without proof. This implies that for the rest of the book whenever we add 3+5 and calculate it to be 8, we know exactly why, 'because of the Addition Table' axiom or '[AT]' for short.

+	1	2	3	4	5	etc
1	2	3	4	5	6	
2	3	4	5	6	7	
3	4	5	6	7	8	
4	5	6	7	8	9	
5	6	7	8	9	10	
etc						

Multiplying in the World of Natural Numbers, $\mathbb{N}$

We will also assume that we know the 'multiplication table' for natural numbers. We will call this the 'times table' axiom, or '[TT]' for short. For the remainder of the book, whenever we need to multiply two natural numbers, like $3 \cdot 5$ we will do so, to obtain 15, and we will justify by stating that the reason we know this to be is because of the Times Tables or [TT].

×	1	2	3	4	5	etc
1	1	2	3	4	5	
2	2	4	6	8	10	
3	3	6	9	12	15	
4	4	8	12	16	20	
5	5	10	15	20	25	
etc						

As stressed in the introduction, we will always celebrate the question 'why'. Thus, if a stranger stopped you in the middle of the street and asked you 'what is 5+3?', you could reply 'that is 8. Not only do i know it's 8, I can also tell you why it is 8.' Then you may continue to explain '3+5 is 8 *because* we adapted the addition table as an axiom (or simply because of [AT])!'

The following examples should help you complete the exercises. Note the abbreviation for Addition Table, [AT], will be used. We will always write the abbreviated justification on a column on the right side.

EXAMPLES

On all of these, calculate, and explain why.

(1) 3+7

Solution:

$$3 + 7 = 10 \qquad \text{AT}$$

(2) $2 + 8$

Solution:

$$2 + 8 = 10 \qquad \text{AT}$$

(3) $1 + 1$

Solution:

$$1 + 1 = 2 \qquad \text{AT}$$

(4) $5 \cdot 5$

Solution:

$$5 \cdot 5 = 25 \qquad \text{TT}$$

(5) $5 \cdot 7$

Solution:

$$5 \cdot 7 = 35 \qquad \text{TT}$$

(6) $3 - 5$

Solution:

$$3 - 5 = ?? \qquad \text{we don't know this yet,}$$

We have not learned to subtract in this class. We will need a bit more tools before we can tackle this difficult problem, be patient.

(7) $5 \div 2$

Solution:

we have not learned to divide yet, patience....

(8) $3 + 0$

Solution:

We have not learned to add 0. We only learned to add natural numbers. Zero is not a natural number. Patience...

Excercises 1.2

(1) $5 \cdot 3$
(2) $5 + 2$
(3) $5 + 0$
(4) $3 + 1$
(5) $32 + 10$
(6) $-1 + 3$
(7) $-1 \cdot -3$
(8) $5 \cdot 0$

1.3. Commutativity Axiom

"To visit the restroom then wash your hands is not the same as washing your hands and then going to the restroom... "

Gameplan 1.3

(1) *what is commutativity*
(2) *what is CoLA*
(3) *what is CoLM*

THE IDEA

The task here is to become aware, familiar and highly sensitive to the idea of commutativity. In layman terms, we can offer the following analogy. To visit the restroom then wash your hands is not the same as washing your hands and then going to the restroom. Clearly, the order in which things are done is sometimes very important. Yet, sometimes the order is which things are done is irrelevant. Suppose you take 2 apples and then take 3 more apples. The net result is you've got 5 apples. Had you taken instead 3 apples then 2 apples, the result would have been identical, 5 apples. In this case, the order is irrelevant. The fancy way to say this is 'the natural numbers are commutative under addition'. If that is too long, the fancy short way of saying it is '$(\mathbb{N}, +)$ is *commutative*'. Keep in mind to determine if an operation is commutative it is necessary to know; first, what the operation is (add, multiply, subtract, etc), second, what set is being operated in. In this section, we meet the world famous commutativity axioms. We will study two such axioms, one for addition and one for multiplication. The Commutativity Law of Addition basically says that we will accept the idea that 2+3 is the same as 3+2. The Commutativity Law of Multiplication, in a nutshell, says that $3 \cdot 2 = 2 \cdot 3$. Of course, the axiom is accepted for any pair of numbers. There is nothing special about 2 and 3.

THE COMMUTATIVITY AXIOMS

(1) Commutativity Law of Addition (CoLA): says that for any numbers A and B,
$$A + B = B + A$$
(2) Commutativity Law of Multiplication (CoLM): says that for any numbers A and B,
$$A \cdot B = B \cdot A$$

EXAMPLES

(1) Is 3+10=10+3? why?
yes! by Co.L.A.
(2) Is $3 \cdot \pi = \pi \cdot 3$? why?
yes! by Co.L.M.

(3) If x is a number and f is a number is $f(x) = x(f)$? why?
yes! by Co.L.M.
(4) If o is a number and n is a number is $n \cdot o = o \cdot n$? why?
yes! by Co.L.M.
(5) If o is a letter and n is a letter is $no = on$? why?
No! Co.L.M is accepted only for numbers not for letter/words
(6) Is $\sin x = x \sin$? why? why not?
Maybe, if each letter represents a number then the equation is true by Co.L.M, but if each letter is not necessary a number then all bets are off, we can not apply Co.L.M.

Excercises 1.3

(1) In your own words, explain the two axioms from this section.
(2) Is 2+5=5+2? Why or why not?
(3) Is $2 \cdot 5 = 5 \cdot 2$? Why or why not?
(4) Suppose x is a number. Is $x \cdot 5 = 5 \cdot x$? Why or why not?
(5) Suppose x and y are numbers. Is $(x + y)5 = 5 \cdot (y + x)$? Why or why not?
(6) Is $is = si$? Why or why not?
(7) Is $\frac{1}{3} + 5 = 5 + \frac{1}{3}$? Why or why not?
(8) Suppose $blah$ is a number and π is a number, is $blah \cdot \pi = \pi \cdot blah$. Why or why not?
(9) Is $\log x = x \log$? Why or why not?
(10) Is $5 \cdot 3 \cdot 7 = 7 \cdot 3 \cdot 5$? Why or why not?
(11) Is $5 + 3 + 7 = 7 + 3 + 5$? Why or why not?
(12) Is $(x + 3) + 7 = 7 + (x + 3)$? Why or why not?

1.4. Associativity Axiom

"For people, associating with different people has different implications. Left side of the equation show two happy couples, the right side show some Joe's wife associating with Mike, the left side is not equal to the right side.... "

Gameplan 1.4
1. *what is associativity*
2. *what is ALA*
3. *what is ALM*

THE IDEA

The task at hand is to become aware and familiar with the Associativity Axioms. There are two of them, one for addition, and one for multiplication. The addition one is usually knows as the Associativity Law of Addition (A.L.A). The multiplication counterpart is know as the Associativity Law of Multiplication (A.L.M.). In layman terms, A.L.A. says if you got 2 or more addition signs, it does not matter which addition you perform first. For example, if we have the quantity '3+5+7', there are a couple of choices here. Should one add the 3 and the 5 first, then add the 7? OR should one add the 5 and 7 first then the 3? or does it matter? A.L.A. says precisely it doesn't matter, as can we can verify, both approaches lead to 15. The technical way to write this is:

$$3 + 5 + 7 = (3 + 5) + 7 = 3 + (5 + 7)$$

Similarly, ALM says when multiplying 3 or more numbers, it does not matter which pair we multiply first. For example:

$$3 \cdot 5 \cdot 7 = 3(5 \cdot 7) = (3 \cdot 5)7$$

A couple of point to keep in mind regarding the associativity axiom.

1. Associativity only rearranges parenthesis. It DOES NOT rearrange the numbers, or quantities. IT only rearranges parenthesis.
2. We will assume the associativity axioms for 'numbers,' under addition or multiplication. There are expressions that are not associative. For example, $\sin(3x)$ is not necessarily the same as $(\sin 3)x$, especially if we don't know if sin is a number.
3. Some operation are not associative. For example, $(8 \div 4) \div 2$ is not the same as $8 \div (4 \div 2)$. Note
$$(8 \div 4) \div 2 = 2 \div 2 = 1$$
while the other is
$$8 \div (4 \div 2) = 8 \div 2 = 4$$
4. Associativity only rearranges parenthesis. It DOES NOT rearrange the numbers, or quantities. IT only rearranges parenthesis.

We summarize below.

The Associativity Axioms

(1) Associativity Law of Addition (A.L.A): says that for any numbers A, B, C and D, we accept

$$(A+B)+C+D = A+(B+C)+D = A+B+(C+D) = A+B+C+D$$

(2) Associativity Law of Multiplication (A.L.M): says that for any numbers A, B, C and D, we accept

$$(AB)CD = A(BC)D = AB(CD) = ABCD$$

Examples

(1) Is $3+(5+7) = (3+5)+7$? why or why not?

Solution:

Yes, by ALA; note the order did not change 3..5...7 on the left and 3...5...7 on the right side

(2) Is $3+y+2+x = 3+(y+2)+x$? why or why not?

Solution:

It depends, if we know x and y are numbers then we can apply the ALA to conclude the equation is true. If we y and x are some not numbers then all bets are off

(3) True or False, explain.

$$(5+3)+(2+10) = 5+(3+2)+10$$

Solution:

Yes, by ALA. Keep in mind ALA says that when we are adding a bunch of numbers, it doesn't matter which ones we associate and add first.

(4) We will abbreviate J for Joe, JW for Joe's wife, M for Mike, and MW for Mike's Wife. True or False, explain.

$$(J+JW)+(M+MW) = J+(JW+M)+MW$$

Solution:

Definitely not. For people, associating with different people has different implications. Left side of the equation show two happy couples, the right side show some Joe's wife associating with Mike, the left side is not equal to the right side.

This points out the fact that the the ALA axiom we have accepted applies to numbers only and it may not apply to other sets (like the set of people).

(5) Is $3(4 \cdot 7) = (3 \cdot 4)7$? Explain.

Solution:

Yes, by ALM.

(6) True or False. Explain.

$$\frac{3}{7} \cdot 3 \cdot (stuff) = (\frac{3}{7} \cdot 3)stuff$$

Solution:

If $'stuff'$ is a number, then yes, equality holds by ALM.

Excercises 1.4

(1) In your own words explain the two axioms from this section.
(2) True or False. Why or Why not. $2 + (3 + 7) = (2 + 3) + 7$
(3) True or False. Why or Why not. $2 \times (3 \times 7) = (2 \times 3) \times 7$
(4) True or False. Why or Why not. $5(7 \cdot 17) = (5 \cdot 7)17$
(5) True or False. Why or Why not. $\pi(3 \cdot \sqrt{2}) = (\pi \cdot 3)\sqrt{2}$
(6) True or False. Why or Why not. $5 + 3 + (5 + 6) + 2 = 5 + 3 + 5 + (6 + 2)$
(7) True or False. Why or Why not. $5 \cdot 3(5 \cdot 6)2 = 5 \cdot 3 \cdot 5(6 \cdot 2)$
(8) True or False. Why or Why not. $x^2 + (3x + 6x) + 18 = (x^2 + 3x) + (6x + 18)$
(9) True or False. Why or Why not. $(\log x)y = \log(xy)$
(10) True or False. Why or Why not. $(\sin x)y = \sin(xy)$
(11) Assuming A, B, C are numbers, then the following statement is true. Explain why (i.e. Prove it!).

$$A + (B + C) = (C + B) + A$$

1.5. The Identities and Inverses

"Surely, you have asked yourself, 'what is the opposite of two?'. Is the opposite of two, owt? or is it -2? or $\frac{1}{2}$?...."

Gameplan 1.5

(1) *what is 'an identity'*
(2) *Additive Identity, [AId]*
(3) *Multiplicative Identity, [MId]*
(4) *what are 'inverses'*
(5) *Additive Inverses, [AInv]*
(6) *Multiplicative Inverses, [MInv]*

The Idea

We continue on quest for the best understanding of addition and multiplication. In this section, we will learn what an identity is and what inverses are.

To learn what an identity is, it may be helpful to review what a binary operation is. Better yet, it may be helpful to think of a binary operation as a mixing beaker, sort of like ones you would use in a chemical lab to mix substances. Imagine a big glass beaker with a big '+' clearly painted on it. Then imagine you reach into your pocket and pull out a '7' and a '3' and throw them into the '+' mixing beaker. You mix it up, shake it up (the beaker) and out comes the result, a '10'. This is very much the way every binary operations works. You throw two things in there mix them according to some recipe either addition, multiplication, subtraction or whatever recipe, and out comes a bran new number. Suppose you throw the 7 and the 3 into the $\times$ beaker. You would get a 21 out of the mixture.

With this analogy in mind we are ready to completely and profoundly understand what an identity is. The identity is the element that acts like water. That is, the element that does not change anyone when you mix it in. For example, if we throw 7 and 0 in the '+' beaker we get a 7. For 5 and 0 into the '+' recipe we get 5. After a while you realize that 0 has absolutely no effect on any other number when mixed with it. In this sense, 0 is like water in the world of addition, it does not change any number under addition. In technical terms, we say '0 is the Additive Identity', and we will accept this as an axiom. We will abbreviate the additive identity axioms as [A.Id.].

On the other hand, in the world of multiplication, if we 'mix' 5 and 1, we get 5, $7\times 1 = 7$, and $1 \times 9 = 9$. Thus, in the $'\times'$ world, 1 acts like water. It does not change any other element when 'multiplied' by 1. The technical way to say this is '1 is the multiplicative identity' and we will accept this as an axiom. The abbreviation for the multiplicative axiom is [M.Id]

We summarize below.

Additive Identity Axiom

For any number a,

$$a + 0 = a \text{ and } 0 + a = a$$

Examples

Calculate and explain why.

(1) $3 + 0$

Solution:

$$3 + 0 = 3 \qquad \text{AId}$$

(2) $0 + 8$

Solution:

$$0 + 8 = 8 \qquad \text{AId}$$

(3) $0 + x + 3$

Solution:

$$0 + x + 3 = x + 3 \qquad \text{AId}$$

Multiplicative Identity Axiom

For any number a,

$$a \cdot 1 = a \qquad \text{and } 1 \cdot a = a$$

Multiplicative Identity Examples

Calculate and Justify

(1) $1(-4)$

Solution:

$$1(-4) = -4 \qquad \text{MId}$$

(2) $1 \cdot x$

Solution:

$$1 \cdot x = x \qquad \text{MId}$$

(3) $1 \cdot \pi$

Solution:

$$1 \cdot \pi = \pi \qquad \text{MId}$$

What are Inverses

You can think about 'inverses' as 'opposites'. Surely, you have asked yourself, 'what is the opposite of two?'. Is the opposite of two, owt? or is it -2? or $\frac{1}{2}$? The point here is that there are many types of opposites, thus there are many types of inverses. Therefore, to be completely understood we will always specify the type of inverse we are talking about. The type of inverse will be given the binary operation. For example, if we ask 'what is the *additive* inverse of 2? The fact that we specified the binary operation narrows down the above choices. The key feature is that *inverses are pairs that when operated together give the identity.* Since 2 added to -2 yields 0 (the add identity), these are *additive inverses.* Since 2 and $\frac{1}{2}$ yield 1 (the multiplicative identity) these are *multiplicative inverses.*

We will accept two more axioms here. One, that every number has a unique inverse. Moreover, we will adopt a convention. Instead of saying 'the additive inverse of a' we could say 'the number that when you add it to a yields 0' or the simply '$-a$'. Moreover, we will accept the idea that every number has a unique additive inverse. In other words, -3 is by definition, the number that when you add it to 3 you get 0. Equivalently, -3 is defined to be the additive inverse of 3. Moreover, there is nothing special about 3, every single number has a unique additive inverse.

Inverse Definitions

(1) **A Pair of Inverses**: is a pair of elements that when operated together yield the identity.
(2) **A Pair of Additive Inverses**: A pair of element that when *added* together yield 0
(3) **A Pair of Multiplicative Inverses**: A pair of element that when *multiplied* together yield 1

Inverse Axioms

(1) (AInv) Every number a has a unique additive inverse denoted $-a$ and read 'negative a'. Thus,

$$a + -a = 0 \text{ and } -a + a = 0$$

(2) (MInv) Every non-zero number a has a unique multiplicative inverse denoted $\frac{1}{a}$. Thus,

$$a \cdot \frac{1}{a} = 1 \text{ and } \frac{1}{a} \cdot a = 1$$

Additive Inverse Examples

(1) $3 + -3$

Solution:

$3 + -3 = 0$ AInv

(2) $5 + -5$

Solution:

$5 + -5 = 0$ AInv

(3) $x + -x$

Solution:

$x + -x = 0$ AInv

(4) $-4 + 4$

Solution:

$4 + -4 = 0$ AInv

Multiplicative Inverse Examples

(1) $5 \cdot \frac{1}{5}$

Solution:

$$5 \cdot \frac{1}{5} = 1 \qquad \text{MInv}$$

(2) $7 \cdot \frac{1}{7}$

Solution:

$$7 \cdot \frac{1}{7} = 1 \qquad \text{MInv}$$

(3) $\frac{1}{8} \cdot 8$

Solution:

$$\frac{1}{8} \cdot 8 = 1 \qquad \text{MInv}$$

(4) $5 \cdot \frac{1}{3}$

Solution:

we have not learned how to multiply these yet... patience

Excercises 1.5

(1) In your own words explain what an identity is.
(2) In your own words explain what is a pair of inverses.
(3) What is the multiplicative inverse of 1?
(4) Calculate and Justify

(a) $5 \cdot 1$
(b) $7 \cdot 1$
(c) $7 \cdot 3$
(d) $7 + 0$
(e) $1 \cdot \frac{2}{3}$
(f) $1 \cdot x$
(g) $0 + x$
(h) $1 \cdot (x + y)$
(i) $(x + y) + 0$
(j) *$3 \cdot 0$

(5) Calculate and Justify

(a) $3 + -3$
(b) $5 + -5$
(c) $-6 + 6$
(d) $7 + -7$
(e) $\sqrt{7} + -\sqrt{7}$
(f) $\frac{3}{7} + -\frac{3}{7}$
(g) $-x + x$
(h) $blah + -blah$
(i) $-cat + cat$
(j) $-(x + 3) + (x + 3)$
(k) $-x + x$
(l) $3 + x$
(m) If $5 + y = 0$ then $y = -5$ True or false?)
(n) If $1 + blah = 0$ then $blah = -1$ (True or false?)
(o) If $-1 + cat = 0$ then $cat = 1$ (True or false?)
(p) If $-5 + cat = 0$ then $cat = 5$ (True or false?)
(q) If $-a + xyz = 0$ then $xyz = a$ (True or false?)

1.6. Distributive Law

"The Distributive Law, *DL, is a very powerful axiom that in some way connects the two operations addition and multiplication....."*

Gameplan 1.6

(1) *What is distributive law*
(2) *DL to remove parenthesis*
(3) *DL to add parenthesis*
(4) *DL to combine terms*
(5) *DL to give us choices*

IDEA

We first learned about the addition axioms, then the multiplication axioms. The *Distributive Law,* DL, is a very powerful axiom that in some way connects the two operations addition and multiplication. Recall, DL says $a(b+c) = ab+ac$ and that $(b+c)a = ba+ca$. Below we list some of the most famous uses for DL. Keep in mind that DL is very fertile. We will keep drawing great results from it for the remainder of the course. For now, we will be content just getting used to it, and learning exactly what it says and doesn't say.

Examples: DL to get Rid of Parenthesis

(1) $3(5+2) = 3 \cdot 5 + 3 \cdot 2$ [DL]
(2) $3(x+y) = 3 \cdot x + 3 \cdot y$ [DL]
(3) $-3(x+y+z) = -3 \cdot x + -3 \cdot y + -3 \cdot z$ [DL]
(4) $(5+7)2 = 5 \cdot 2 + 7 \cdot 2$ [DL]
(5) $\pi(2+\sqrt{5}) = \pi 2 + \pi\sqrt{5}$ [DL]
(6) $-1(a+-a) = -1 \cdot a + -1 \cdot -a$ [DL]
(7) $stuff(x+3) = stuff \cdot x + stuff \cdot 3$ [DL]
(8) $cat(x+3) = cat \cdot x + cat \cdot 3$ [DL]
(9) $(2+4)(x+3) = (2+4)x + (2+4)3$ [DL]
(10) $(2+4)(x+3) = 2(x+3) + 4(x+3)$ [DL]

Examples: DL to Add Parenthesis (factor)

(1) $3 \cdot 5 + 3 \cdot 7 = 3(5+7)$ [DL]
(2) $-1 \cdot 1 + -1 \cdot -1 = -1(1+-1)$ [DL]
(3) $x \cdot 5 + x \cdot 7 = x(5+7)$ [DL]
(4) $3\sin t + 5\sin t = (3+5)\sin t$ [DL]
(5) $3\log t + 5\log t = (3+5)\log t$ [DL]
(6) $3(x+23) + 5(x+23) = (3+5)(x+23)$ [DL]
(7) $5\sqrt{7} + x\sqrt{7} = (5+x)\sqrt{7}$ [DL]
(8) $bla \cdot 5 + bla \cdot 7 = bla(5+7)$ [DL]
(9) $(\mathbf{1}+\mathbf{3})5 + (\mathbf{1}+\mathbf{3})7 = (\mathbf{1}+\mathbf{3})(5+7)$ [DL]

(10) $xy \cdot 3 + xy \cdot r = xy(3 + r)$ [DL]
(11) $red + rest = re(d + st)$ [DL]
(12) $h\sqrt{5} + h\pi = h(\sqrt{5} + \pi)$ [DL]

Examples: DL to Combine Like Terms

(1) Combine $3x + 5x$

$$\begin{aligned} 3x + 5x &= (3+5)x && \text{DL} \\ &= 8x && \text{AT} \end{aligned}$$

(2) Combine $3y + 10y$

$$\begin{aligned} 3y + 10y &= (3+10)y && \text{DL} \\ &= 13y && \text{AT} \end{aligned}$$

(3) Combine $3\log 5 + 10\log 5$

$$\begin{aligned} 3\log 5 + 10\log 5 &= (3+10)\log 5 && \text{DL} \\ &= 13\log 5 && \text{AT} \end{aligned}$$

(4) Combine $3\sqrt{5} + 10\sqrt{5}$

$$\begin{aligned} 3\sqrt{5} + 10\sqrt{5} &= (3+10)\sqrt{5} && \text{DL} \\ &= 13\sqrt{5} && \text{AT} \end{aligned}$$

Examples: DL to Give us Choices

The Distributive Law basically gives us some nice choices. Consider for example the quantity $4(5+3)$. If we wanted to simplify this quantity we could: think $5 + 3 = 8$, so we can substitute and conclude $4(5+3) = 4 \cdot 8 = 32$. On the other hand the distributive law gives us another choice for carrying out the calculation. The Distributive Law says: $4(5+3) = 4 \cdot 5 + 4 \cdot 3$, now we substitute $4 \cdot 5 = 20$ and $4 \cdot 3 = 12$ ... so $4(5+3) = 4 \cdot 5 + 4 \cdot 3 = 20 + 12 = 32$. Put another way, the distributive law says you can either add the stuff inside the parenthesis first then multiply OR get rid of the parenthesis first by using the Distributive Law Recipe then add the stuff at the end.

(1) Simplify $4(5+2)$ with and without using D.L.

Without D.L.

$$\begin{aligned} 4(5+2) &= 4(7) && \text{AT} \\ &= 28 && \text{TT} \end{aligned}$$

Using D.L.

$$\begin{aligned} 4(5+2) &= 4\cdot 5 + 4\cdot 2 && \text{DL} \\ &= 20+8 && \text{TT} \\ &= 28 \end{aligned}$$

(2) Simplify $4(1+3)$ with and without using D.L.

Without D.L.

$$\begin{aligned} 4(1+3) &= 4(4) && \text{AT} \\ &= 16 && \text{TT} \end{aligned}$$

Using D.L.

$$\begin{aligned} 4(1+3) &= 4\cdot 1 + 4\cdot 3 && \text{DL} \\ &= 4+12 && \text{M.Id, TT} \\ &= 16 \end{aligned}$$

Excercises 1.6

(1) Determine if statement is True of False. If it is true, provide appropriate justification.

(a) $3(5+2) = 3\cdot 5 + 3\cdot 2$
(b) $5\cdot\frac{1}{3} + 2\cdot\frac{1}{3} = (5+2)\frac{1}{3}$
(c) $3(5\cdot 2) = 3\cdot 5\cdot 3\cdot 2$
(d) $5\sqrt{3+7} = \sqrt{5\cdot 3} + \sqrt{5\cdot 7}$
(e) $(x+3)5 = x5 + 3\cdot 5$
(f) $(x+3)bla = x\cdot bla + 3bla$
(g) $x\cdot stf + 3\cdot stf = (x+3)stf$
(h) $x(2+y) + t(3+y) = (x+t)(2.5+y)$
(i) $x(x+2) + 5(x+2) = (x+5)(x+2)$
(j) $2(\sqrt{5}+x) = 2\sqrt{5} + 2x$

(2) Get rid of all parenthesis **Using DL**. Calculate when possible. Show, Justify and Understand Every step.

(a) $5(3+2)$
(b) $(5+2)7$
(c) $4(2+5+-7)$
(d) $4(2+5+x)$
(e) $(4+x)(3+y)$
(f) $(2+3)(5+1)$

(3) Use DL to factor out common terms. Simplify if possible.

(a) $5x+4x$
(b) $5x+-4x$
(c) $3\sqrt{5}+8\sqrt{5}$
(d) $4\sin x + 7\sin x$
(e) $(4+x)3 + (4+x)8$
(f) $3+2\sqrt{5}$
(g) $3\sqrt{5}+2\sqrt{5}$
(h) $3(x+4)+2(x+4)$
(i) $3(x+4)+-y(x+4)$
(j) $3(x+4)+y(x+5)$
(k) $3x^2+5x^2$
(l) $\sqrt{4}\sin x + \pi\sin x$

(4) Perform the following Using DL and again without using DL.

(a) $5(3+2)$
(b) $5(1+2)$
(c) $(2+7)2$
(d) $2(3+5)$

1.7. Equality Axioms

"a rose is a rose is a rose... Reflexive property"

Gameplan 1.7
(1) *Reflexive Property*
(2) *Symmetric Property*
(3) *Transitive Property*
(4) *Cancellation Law of Addition*
(5) *Cancellation Law of Multiplication*

IDEA

Here we will become very aware of what we can and can't do to both sides of an equation. We first assume list several important axioms. For example, we will assume that to any true equation we can always add any number to both sides and still have a true equation. This is called the Cancellation Law of Addition (CLA). The Cancellation Law of Multiplication (CLM) say we can always multiply both sides of an equation by any number we want. We summarize these and other famous *equality axioms*.

EQUALITY AXIOMS

Name	Symbol	Means...
Reflexive Property	[RP]	$a = a$
Symmetric Property	[SP]	If $a = b$ then $b = a$
Transitive Property	[TP]	If $a = b$ and $b = c$ then $a = c$
Cancellation Law of Addition	[CLA]	If $a = b$ then $a + c = b + c$
Cancellation Law of Multiplication	[CLM]	If $a = b$ then $ac = bc$

PRACTICE RP

The reflexive property says any number is equal to itself. In other words, because of RP we can safely assume 3=3, or 5=5, or a rose is a rose is a rose... This is so obvious it is a classic text-book example of an axiom.

(1) $3 = 3$ [True by Reflexive Property]
(2) $5 = 5$ [RP]
(3) $a\,rose = a\,rose$ [RP]
(4) $3 = x$ [may be true, but it's not what RP says]

PRACTICE SP

The symmetric property says that the quantities on either side of an equal sign can be interchanged and the equality will still hold. Using math symbols, if $a = b$ then $b = a$. Using English words, if Rob= Robert, then "Robert= Rob."

(1) If $3 = x$ then $x = 3$ [True by Symmetric Property]
(2) If $5 = blah$ then $blah = 5$ [SP]
(3) If $3x + 2 = y$ then $y = 3x + 2$ [SP]
(4) if $3\pi = x$ then $x = 3\pi$ [SP]
(5) if *blue* + *yellow* = *green* then *green*= *blue*+*yellow*

PRACTICE TP

The transitivity property says if a first quantity equals a second quantity, and the second quantity equals a third quantity, then the first quantity must be equal to the third quantity. Using math symbols, if $a = b$ and $b = c$ then $a = c$.

(1)

$$\begin{aligned} \text{if } 3 &= x \\ \text{and } x &= y \\ \text{then } 3 &= y \qquad \text{true by transitivity property} \end{aligned}$$

(2)

$$\begin{aligned} \text{if } 3 &= 3 \cdot 1 \\ &= 3 \cdot (-1)(-1) \\ \text{then } 3 &= 3 \cdot (-1)(-1) \qquad \text{true by TP} \end{aligned}$$

PRACTICE CLA

Cancellation Law of Addition says we can always add any quantity to both sides of an equation, and the equation will remain valid, so long as we add the same quantity to both sides.

(1)

$$\begin{aligned} \text{if } y &= x \\ \text{then } 3 + y &= 3 + x \qquad \text{True, added 3 to both sides,CLA} \end{aligned}$$

(2)

$$\begin{aligned} \text{if } y + 2 &= x - 7 \\ \text{then } (y + 2) + x &= (x - 7) + x \qquad \text{CLA, added } x \end{aligned}$$

(3)

$$\text{if } 3x + 2 = 5x$$
$$\text{then } 3x + 2 + -2 = 5x + -2 \qquad \text{CLA, added -2}$$

(4)

$$\text{if } 3x + 2 = 5x$$
$$\text{then } 3x + 2 + 5 = 5x + 5 \qquad \text{CLA}$$

(5)

$$\text{if } 3x + 2 = 5x$$
$$\text{then } 3x + 2 + \pi = 5x + \pi \qquad \text{CLA}$$

PRACTICE CLM

Cancellation Law of Multiplication says we can always multiply both sides of an equation by any quantity, and the equation will remain valid, so long as we multiply both sides by the same quantity.

(1)

$$\text{if } y = x$$
$$\text{then } 3y = 3x \qquad \text{True, Mult both sides by 3, CLM}$$

(2)

$$\text{if } y + 2 = x - 7$$
$$\text{then } (y + 2)x = (x - 7)x \qquad \text{CLM, Mult by } x$$

(3)

$$\text{if } 3x + 2 = 5x$$
$$\text{then } (3x + 2)(\frac{1}{5}) = (5x)(\frac{1}{5}) \qquad \text{CLA, Mult by 1/5}$$

(4)

$$\text{if } 3x + 2 = 5x$$
$$\text{then } \frac{1}{3} \cdot 3x + 2 = \frac{1}{3} \cdot 5 \qquad \text{False, no such axiom}$$

(5)

$$\text{if } 3x+2 = 5x$$
$$\text{then } \frac{1}{3}(3x+2) = \frac{1}{3}\cdot 5 \qquad \text{CLM}$$

Excercises 1.7

Determine if the statement is True or False. If true, provide appropriate justification.

(1) $5 = 5$
(2) $a = a$
(3) $0 = 0$
(4) If $4 = x + \pi$ then $x + \pi = 4$
(5) If $4 = x + \pi$ and $x + \pi = t$ then $4 = 7$
(6) If $4 = x + \pi$ then $\frac{1}{3} \cdot 4 = \frac{1}{3} \cdot x + \pi$
(7) If $4 = x + \pi$ then $\frac{1}{3} \cdot 4 = \frac{1}{3}(x + \pi)$
(8) If $4 = 3x + 2$ then $\frac{1}{3} \cdot 4 = \frac{1}{3} \cdot 3x + 2$
(9) If $4 = 3x + 2$ then $\frac{1}{3} \cdot 4 = \frac{1}{3}(3x + 2)$
(10) $(x + y)7 = 7(x + y)$
(11) If $4 = x + \pi$ then $\pi + x = 4$
(12) If $4 = x + 2$ then $x = 2$
(13) $5 = \sqrt{t} + 7$ and $\sqrt{t} + 7 = 2$ then $5 = 2$

1.8. Exponents

"We continue on our quest for the freshest point of view, this time on exponents..."

Gameplan 1.8
(1) *What are Exponents*
(2) *0-Exponent*
(3) *Natural Exponents*
(4) *Negative Exponents*

Idea

We continue on our quest for the freshest point of view, this time on exponents. We pretend we have never seen an exponent before and we ask 'what exactly is an exponent?' Below we take a moment to consider various types of exponents and their respective meaning.

Natural Number Exponents

Suppose we wanted to express the quantity, $3 \cdot 3 \cdot 3 \cdot 3$, in various ways. One obvious alternative is to carry out the multiplication, and rather than writing $3 \cdot 3 \cdot 3 \cdot 3$, we can simply write 81.

Exponents give us a second alternative. Rather than writing $3 \cdot 3 \cdot 3 \cdot 3$, we can simply write

$$3^4$$

In this case the 4 is called the 'exponent', while the 3 is called the 'base'. The exponent, '4', next to the base, '3' simply means we are too lazy to write the base '3' four times as in $3 \cdot 3 \cdot 3 \cdot 3$. In other words, an exponent of '4' on any base simply means the product of the base times itself 4 times.

The following examples illustrate the definition of Natural Number Exponents, abbreviated *[N-Expo].*

Examples: Definition of Natural Exponents

(1) rewrite 3^5

Solution:

$$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \qquad \text{N-Expo}$$

(2) rewrite x^2

Solution:

$$x^2 = x \cdot x \qquad \text{N-Expo}$$

(3) rewrite $(blah)^2$

Solution:

$$(blah)^2 = (blah)(blah) \qquad \text{N-Expo}$$

(4) rewrite $blah^2$

Solution:

note in this case, the exponent is still 2 but the base is not the entire 'blah'. The base is always whatever is imediately next to the exponent. If there are patenthesis next to the exponent then that is the base. If there are no parenthesis the base is only the stuff next to the parenthesis. In this case, since there is no parenthesis, the base is just the $'h'$. We can rewrite as:

$$blah^2 = blah \cdot h \qquad \text{N-Expo}$$

(5) rewrite $(5x)^3$

Solution:

$$(5x)^3 = (5x)(5x)(5x) \qquad \text{N-Expo}$$

(6) rewrite $5x^3$

Solution:

$$5x^3 = 5x \cdot x \cdot x \qquad \text{N-Expo}$$

(7) rewrite $(x+1)^2$

Solution:

$$(x+1)^2 = (x+1)(x+1) \qquad \text{N-Expo}$$

ZERO EXPONENT

The definition above will help us understand what we mean when we write expressions such as 3^4. Specifically, we mean write 3 times itself four times. Then, consider a sensible way to interpret 3^0. The logic used to define Natural Number Exponents would suggest 'write 3 times itself 0 times'. How do we write something times itself 0 times? Well, we don't. Instead we will adapt a different definition for the 0 exponent. We will define,

$$3^0 = 1$$

Of course there is nothing special about 3. We will take the 0-Exponent to mean the same on every non-zero base, a. Namely, if $a \neq 0$, we define Zero Exponent, [0-Expo], by

$$a^0 = 1 \qquad [\text{0-Expo}]$$

NEGATIVE EXPONENTS

We focus on negative exponents. First we define, the exponent -1. When we write 3^{-1} we mean 'the multiplicative inverse of 3'. In other words,

$$3^{-1} = \frac{1}{3}$$

Then we can define other negative powers, by the following convention:

$$3^{-4} = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$$

EXAMPLES: NEGATIVE EXPONENTS

(1) rewrite $(5x)^{-3}$

Solution:

$$(5x)^{-3} = \frac{1}{5x} \cdot \frac{1}{5x} \cdot \frac{1}{5x} \qquad \text{Neg-Expo}$$

(2) rewrite 2^{-3}

Solution:

$$2^{-3} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \qquad \text{Neg-Expo}$$

(3) rewrite 3^{-2}

Solution:

$$3^{-2} = \frac{1}{3} \cdot \frac{1}{3} \qquad \text{Neg-Expo}$$

Excercises 1.8
Rewrite each expression using the definitions of exponents.

(1) 5^4
(2) 2^{-3}
(3) $2x^2$
(4) $(2x)^2$
(5) $(2+x)^2$
(6) $3(x+6)^3$
(7) $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
(8) $(x+1)^{-2}$

1.9. Paperplication

"Here we gather all our might and courage to question the unquestionable..."

Gameplan 1.9
(1) *Idea*
(2) *The Elements*
(3) *The Binary Operations*
(4) *The Famous Questions*

Idea

Math is all about tickling the brain cells, and this section has tickling all over it. Here we gather all our might and courage to question the unquestionable. Could we multiply other things besides numbers? Are there other binary operations besides the same old ones, *addition, subtraction, multiplication, and division*? Could we *invent* a new operation on a new set of elements (non-numbers)? And if we did invent a new operation could we have an identity, like we do for addition ('0') and multiplication ('1'). Do we have inverses, commutativity, or associativity laws?

Indeed we fearlessly charge ahead and 'invent' a new binary operation on a brand new set.

The Set

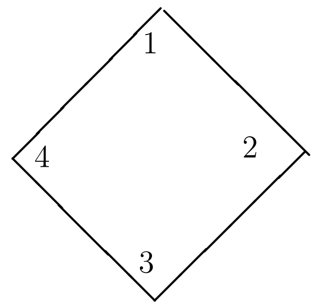
The name of our set will be the *Dihedral Group,* abbreviated D_4. This set has exactly 8 members in it. Already it is very different than the natural numbers, which contain infinite many numbers. The following is a complete list of the 8 elements.

$$D_4 = \{\odot, ↰, ↱, \curvearrowright, |, -, \diagdown, \diagup\}$$

These are the 8 things we are going to 'paperply.' Which means we will figure out a way to take two of them, then paperply them to get a third element. In fact, we should be able to write down a Paperplication table, just as we have written a multiplication and addition table for Naturals.

The Binary Operation: Paperplication

The operation is inspired by a small piece of paper cut into a square shape. Each corner is numbered 1 through 4.



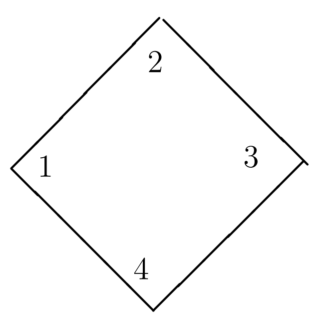
The Paper Behind Paperplication

You will need to construct such a devise. Number the corners as illustrated above, then number the back so that each corner has the same number both in front and in back.

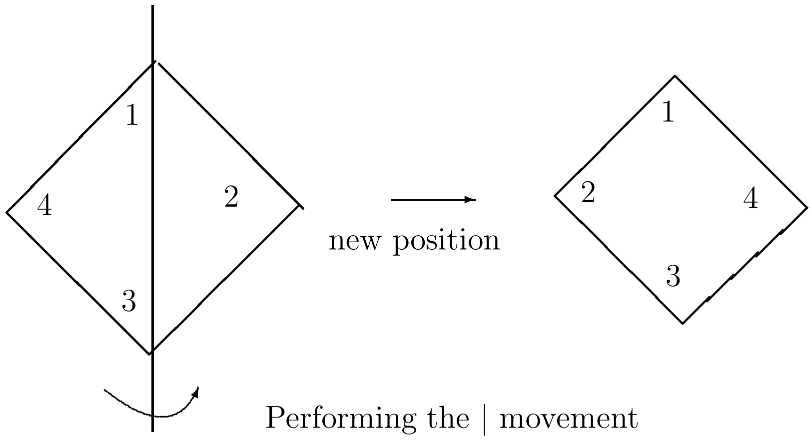
We now take a moment to explain what each of the symbols in D_4 means. Each of the symbols represents a movement on the paper. The drawn position is called standard position, and that is where we will always start before performing any of the operations.

The symbol ↰ means rotate the square 90° counterclockwise. In other words, if ↰ is performed from standard position, we would have the new position

Performing ↰

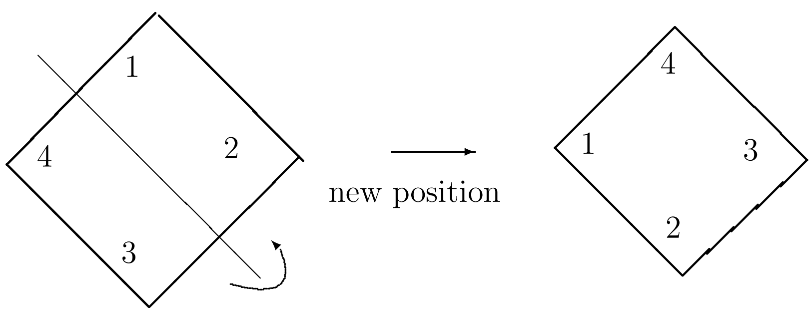


Similarly ↱ means rotate clockwise 90° while ↷ means to rotate clockwise 180°. The straight lines, on the other hand, represent flips. Each one represents a flip about that axis. The vertical line, |, for example represents the following flip:



Performing the | movement

Similarly, the other lines are just flips. Take the diagonal down symbol, $\diagdown$, for example. It says to flip the paper about that axis, as follows..



Performing the $\diagdown$ movement

The only element left to explain is the $\odot$ element. This element means not movement at all. You will find that this element is the identity under paperplication.

Once we understand what each symbol is we are ready to start to 'paperply'. We will use the symbol '$\bigstar$' to mean paperply. The recipe will be:

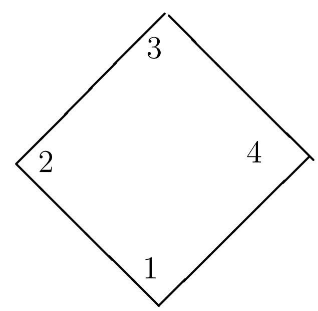
(1) Start in standard position.
(2) Perform the first movement.
(3) Perform the second movement, this is called the result position.
(4) Record the result position.
(5) Rest, take a break.
(6) Now try to get to the result position, with just one move, from the standard position. Whichever movement is required is indeed the paperplication of the two movements.

EXAMPLES: PAPERPLICATION

(1) Calculate $\Rsh \bigstar \Rsh$

Solution:

We first start with standard position, rotate once 90°, then rotate again 90°, then we record the resulting position which is



The Paper Behind Paperplication

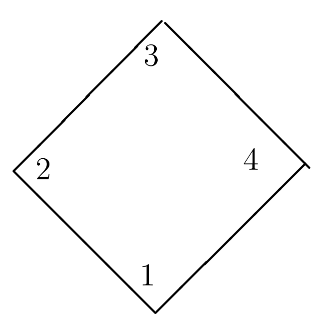
Then we have to think, which movement can get us to the resulting position, in just one move from standard position? After a couple of seconds of thinking (or trying all 8 possibilities) we realize that we can achieve that position with a 180° turn, ↷ . Therefore we can conclude,

↱ ★ ↱ = ↷

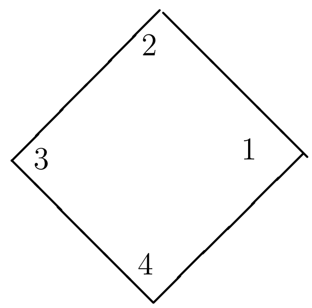
(2) Calculate ↷ ★ ╲

Solution:

Again, we follow the recipe. First we start in standard position. Then we perform ↷ to obtain



From here, we perform ╲ (flip) to obtain



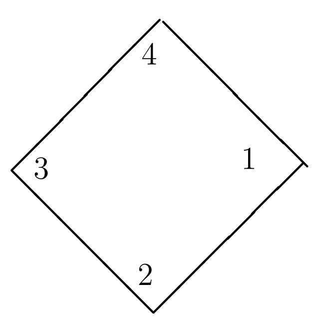
This is the result position. We relax now. Then we have to determine which movement will get us there (result position) in just one move. After thinking about it we conclude ╱ will give us the same position as the result position. So we conclude..

↷ ★ ╲ = ╱

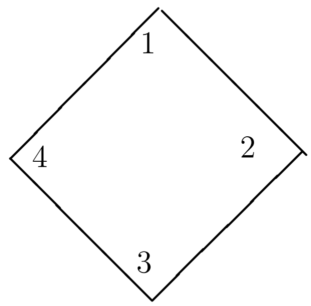
(3) Calculate ↱ ★ ↰

Solution:

Again, we follow the recipe. First we start in standard position. Then we perform a 90° clockwise rotation, ↱, to obtain



From here, we perform ↰ (rotate 90° counterclockwise) to obtain:



This is the result position. We relax now. Then we have to determine which movement will get us there (result position) in just one move. After thinking about it we conclude $\odot$ (no movement!) will give us the same position as the result position. So we conclude..

$$\Rsh \bigstar \Lsh = \odot$$

(1) Commutativity
(2) The Paperplication Table
(3) Associativity
(4) Identity
(5) Inverses

Excercises 1.9
Rewrite each expression using the definitions of exponents.

(1) Calculate
   (a) $\Lsh \bigstar \diagdown$
   (b) $\diagdown \bigstar \curvearrowright$
   (c) $\curvearrowright \bigstar \diagdown$
   (d) Is D_4 commutative under $\bigstar$?

(2) Calculate
   (a) $\odot \bigstar \curvearrowright$
   (b) $\odot \bigstar \diagup$
   (c) $| \bigstar \odot$
   (d) Does D_4 have an identity element under $\bigstar$?

(3) Complete the $\bigstar$ Table for D_4. To complete the table, always use the convention; Row first Column second. For example, since $\curvearrowright \bigstar \diagdown = \diagup$, we find the $\curvearrowright$row, then the $\diagdown$column and we write the product $\diagup$. Some of the table has already been filled. Complete it.

$\bigstar$	$\odot$	$\Lsh$	$\Rsh$	$\curvearrowright$	$\vert$	$-$	$\diagdown$	$\diagup$
$\odot$	$\odot$	$\Lsh$	$\Rsh$	$\curvearrowright$	$\vert$	$-$	$\diagdown$	$\diagup$
$\Lsh$								
$\Rsh$								
$\curvearrowright$							$\diagdown$	
$\vert$								
$-$								
$\diagdown$								
$\diagup$								

(4) Remember, inverses are pairs that yield the identity. In this case, the identity is $\odot$, therefore, inverses are pairs whose product is $\odot$. Find the inverse for each.

(a) $\Rsh$
(b) $\curvearrowright$
(c) $\diagdown$
(d) $\odot$

(5) (Bonus) Solve the equation for x,

$$\diagdown \bigstar x \bigstar \curvearrowright = \Rsh$$

1.10. Chapter Review

Famous Sets

Name	Symbol	Means...
Natural Numbers	$\mathbb{N}$	$\{1, 2, 3, \dots\}$
Whole Numbers	$\mathbb{W}$	$\{0, 1, 2, 3, \dots\}$
Integer Numbers	$\mathbb{Z}$	$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
Rational Numbers	$\mathbb{Q}$	$\{\frac{integer}{integer \neq 0}\}$

Addition Axioms

Comm. Law of Addition	[CoLA]	$a + b = b + a$
Associativity Law	[ALA]	$a + b + c + d = (a + b) + c + d =$ $a + (b + c) + d = a + b + (c + d)$
Additive Identity is 0	[AId]	$a + 0 = 0 + a = a$
Additive Inverses	[AInv]	a and b called *Additive Inverses* If and only if $a + b = 0$. In this case, b is called $-a$.
Distributive Law	[DL]	$a(b + c) = ab + ac$ and $(b + c)a = ba + ca$

Multiplication Axioms

Commutativity Law of Multipl.	[CoLM]	$ab = ba$
Associativity Law	[ALM]	$abcd = (ab)cd = a(bc)d = ab(cd)$
Multiplicative Identity is 1	[M.Id]	$a \cdot 1 = 1 \cdot a = a$
Multiplicative Inverses	[M.Inv]	a and b called *M. Inverses* If and only if $ab = 1$. If so, b is called $\frac{1}{a}$ or a^{-1}.
Fractions	[Def of $\frac{a}{b}$]	$\frac{a}{b} = a \cdot \frac{1}{b} = \frac{1}{b} \cdot a$ $\frac{a}{b}$ has meaning only if $b \neq 0$

Equality Axioms

Reflexive Property	[RP]	$a = a$
Symmetric Property	[SP]	If $a = b$ then $b = a$
Transitive Property	[TP]	If $a = b$ and $b = c$ then $a = c$
Cancellation Law of Addition	[CLA]	If $a = b$ then $a + c = b + c$
Cancellation Law of Addition	[CLM]	If $a = b$ then $ac = bc$

Excercises 1.10

For 1-53, if statement is true, name the axiom or definition, otherwise write 'false.' For 54-58, calculate showing every step.

(1) If $a = 3$, and $3 = c$, then $a = c$
(2) $\clubsuit + (\rho + \xi) = (\clubsuit + \rho) + \xi$
(3) $cat \cdot (cat)^{-1} = 0$
(4) IF $x = y$
Then $x(3 - v) = y(3 - v)$
(5) $(3 \cdot 5)2 = 3(5 \cdot 2)$
(6) $x \cdot \frac{1}{x} = 1$
(7) $-x + x = 0$
(8) $-(blag) + blag = 0$
(9) $blah \cdot \frac{1}{blah} = 1$
(10) $-(x + y - 4) + (x + y - 4) = 0$
(11) If $1 + cat = 0$
then $cat = -1$
(12) If $1 + joe = 0$ then $joe = -1$
(13) IF $1 + (-1)(-1) = 0$
then $(-1)(-1) = -1$
(14) If $-1 + (-1)(-1) = 0$
then $(-1)(-1) = 1$
(15) If $5 \cdot cat = 1$
then $cat = \frac{1}{5}$
(16) If $15 \cdot x = 1$
then $x = \frac{1}{15}$
(17) If $8 + b = 0$ then $b = -8$
(18) If $8 + blah = 0$
then $blah = -8$
(19) If $8 + (-5 + -3) = 0$
then $(-5 + -3) = -8$
(20) $\clubsuit\heartsuit = \heartsuit\clubsuit$
(21) $\heartsuit = \heartsuit$
(22) $joe \cdot (joe)^{-1} = 1$
(23) $-\Sigma + \Sigma = 0$
(24) $cat \cdot cat \cdot cat \cdot cat = (cat)^4$
(25) $(x + y)^2 = (x + y)(x + y)$
(26) $(x + y)^3 = (x + y)(x + y)(x + y)$
(27) $(x + y)^3 = x^3 + y^3$
(28) $(x + y)^2 = x^2 + y^2$
(29) $(3 + 5)^2 = 3^2 + 5^2$
(30) $(3 + 5)^2 = (3 + 5)(3 + 5)$
(31) $blah \cdot \frac{1}{dlah} = 1$
(32) $blah \cdot \frac{1}{Blah} = 1$
(33) $X = X$
(34) $\clubsuit = \clubsuit$
(35) $\Gamma(\Sigma + \zeta) = \Gamma\Sigma + \Gamma\zeta$
(36) $3 + 5 = 5 + 3$
(37) If $T + Z - 3 = K + 3R$ then
$T + Z - 3 + (-8) = K + 3R + (-8)$
(38) $\sin x \cdot \frac{1}{\sin x} = 1$
(39) $\cos 3x = 3 \cos x$
(40) $-(x + y - 4) + (x + y - 4) = 0$
(41) $0 + \epsilon = \epsilon$
(42) If $a = b$ then $a + c = b + c$
(43) $(3 \cdot 5)2 = 3(5 \cdot 2)$
(44) $\clubsuit + \heartsuit = \heartsuit + \clubsuit$
(45) If $3 = x$, then $x = 3t$
(46) $(3 \cdot 5)2 = 3(5 \cdot 2)$
(47) $\log x = x \log$
(48) $on = no$
(49) $dear = read$
(50) $3(5 + 4) = 3 \cdot 5 + 3 \cdot 4$
(51) $3(5 \cdot 4) = 3 \cdot 5 \cdot 3 \cdot 4$
(52) $cat(big + bear) = cat \cdot big + cat \cdot bear$
(53) $cat(big \cdot bear) = cat \cdot big \cdot cat \cdot bear$
(54) $-3 + 7$
(55) $-6 + 10$
(56) $10 \cdot \frac{1}{2}$
(57) $-2 + 20 \cdot \frac{1}{5}$
(58) $-2 + 7 + -1$

CHAPTER 2

The Integers

2.1. Multiplying Integers

"All this comes from the axioms, not from a calculator, not because the book said so, and not because your grandmother said so...... "

Gameplan 2.1

(1) *What are Integers*
(2) *Basic Multiplying*
(3) *Multiplying by 0 [0MT]*
(4) *Multiplying Negatives [NNT]*
(5) *Practice*

IDEA

Here we review the process to multiply the integers. Emphasis will be placed on the axioms that make each step irreproachable! We will let the axioms lead us to the right solution. Note also that below we will encounter our very first batch of fruits from the axioms. Namely we will derive a couple of very famous theorems you may already be familiar with. For example, from the axioms, we will derive the fact that any number times zero is zero, or that a negative times a negative is a positive. All this comes from the axioms, not from a calculator, not because the book said so, and not because your grandmother said so....

WHAT ARE INTEGERS

If we take the natural numbers, $\mathbb{N}$, throw in '0', and all the negative counterparts to the naturals, then we have *the integers.* The symbol for integers is $\mathbb{Z}$. We can summarize this definition by:

$$\text{Integers} = \mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3...\}$$

In this chapter we will study all that is *integers,* how to add, subtract, multiply, factor, and more. We start with a few examples multiplying Natural numbers and making use of the definition of exponents.

EXAMPLES: BASIC MULTIPLYING

(1) Example: Calculate $(3^2)\cdot 5$

$$\begin{aligned}(3^2)\cdot 5 &= (3\cdot 3)\cdot 5 && \text{N-Expo}\\ &= (9)(5) && \text{T.T.}\\ &= 45 && \text{T.T.}\end{aligned}$$

(2) Example: Rewrite $3\cdot 3\cdot 5\cdot 7\cdot 3\cdot 5$ using exponent.

$$\begin{aligned}3\cdot 3\cdot 5\cdot 7\cdot 3\cdot 5 &= 3\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7 && \text{C.L.M}\\ &= 3^3\cdot 5^2\cdot 7 && \text{N-Expo}\end{aligned}$$

(3) Example: Calculate $3\cdot(2^3)$

$$\begin{aligned}3\cdot(2^3) &= 3\cdot(2\cdot 2\cdot 2) && \text{Def of Expo}\\ &= 3\cdot 8 && \text{T.T.}\\ &= 24 && \text{T.T.}\end{aligned}$$

Multiplying by 0: The Zero Multiplication Theorem[0MT]

As promised here we deliver the tight reasoning and infallible logic to explain and understand why any number times 0 is 0. To complete the proof we will first see why $3\cdot 0 = 0$. The exact same sort of argument could be used to prove that any number times 0 is 0. Therefore, after we see how it works for 3, we prove it for a generic number $'n'$.

The Example: Don't be afraid to ask: why is $3\cdot 0 = 0$?

$$\begin{aligned}0+0 &= 0 && \text{A.Id}\\ 3(0+0) &= 3\cdot 0 && \text{CLM}\\ 3\cdot 0 + 3\cdot 0 &= 3\cdot 0 && \text{DL}\\ (3\cdot 0+3\cdot 0) + -3\cdot 0 &= 3\cdot 0 + -3\cdot 0 && \text{CLA}\\ 3\cdot 0 + (3\cdot 0 + -3\cdot 0) &= (3\cdot 0 + -3\cdot 0) && \text{ALA}\\ 3\cdot 0 + 0 &= 0 && \text{A Inv}\\ 3\cdot 0 &= 0 && \text{A.Id}\end{aligned}$$

The Theorem: *For any number n,*

$$n\cdot 0 = 0\cdot n = 0$$

Proof:

$$\begin{aligned}
0+0&=0 && \text{A.Id}\\
n(0+0)&=n\cdot 0 && \text{CLM}\\
n\cdot 0+n\cdot 0&=n\cdot 0 && \text{DL}\\
n\cdot 0+n\cdot 0+-n\cdot 0&=n\cdot 0+-n\cdot 0 && \text{CoLA}\\
n\cdot 0+(n\cdot 0+-n\cdot 0)&=n\cdot 0+-n\cdot 0 && \text{ALA}\\
n\cdot 0+0&=0 && \text{A.Inv}\\
n\cdot 0&=0 && \text{A.Id}\\
0\cdot n&=0 && \text{CoLM}
\end{aligned}$$

Thus $n \cdot 0 = 0$ and $0 \cdot n = 0$ for any integer n

The Minus Theorem [MT]

This theorem does not usually get its due respect, yet we will make heavy use of it. The *minus theorem [MT]* says we can always stick a '1' between any number a and a negative before it. That is:

$$-3 = -1 \cdot 3$$

Another way to think about [MT], perhaps the professional way to look it, is read what is say as

the additive inverse of 3 = add inverse of 1 times 3.

To understand the proof we will first strive to understand it on a particular number. We will first prove $-3 = -1 \cdot 3$. Once we understand the proof for -3 we will follow the same logic to prove it works for any generic number $-a$.

The Example: $-3 = -1 \cdot 3$

Here's *why:*

$$\begin{aligned}
1+-1&=0 && \text{A.Inv}\\
(1+-1)3&=0\cdot 3 && \text{CLM}\\
1\cdot 3+-1\cdot 3&=0\cdot 3 && \text{DL}\\
3+-1\cdot 3&=0\cdot 3 && \text{M.Id}\\
3+-1\cdot 3&=0 && \text{0MT}\\
-3+3+-1\cdot 3&=-3+0 && \text{CLA}\\
0+-1\cdot 3&=3+0 && \text{AInv}\\
-1\cdot 3&=-3 && \text{A.Id}
\end{aligned}$$

The Theorem [MT]: *For any number a,*

$$-a = -1 \cdot a$$

Proof:

$$
\begin{aligned}
1+-1&=0 && \text{A.Inv}\\
(1+-1)a&=0\cdot a && \text{CLM}\\
1\cdot a+-1\cdot a&=0\cdot a && \text{DL}\\
a+-1\cdot a&=0\cdot a && \text{M.Id}\\
a+-1\cdot a&=0 && \text{0MT}\\
-a+a+-1\cdot a&=-a+0 && \text{CLA}\\
0+-1\cdot a&=a+0 && \text{AInv}\\
-1\cdot a&=-a && \text{A.Id}
\end{aligned}
$$

NEGATIVE TIMES NEGATIVE THEOREM [NNT]

An all-time favorite classic question for the knowledge thirsty, *'why is* $-1\cdot-1=1$*?'*. A beautiful question deserves nothing less than a beautiful answer. Indeed, the axioms will lead us there. Again we start with an example to illustrate why $-1\cdot-1=1$. We will then proceed to show the same persuasive logic will work for generic numbers $-a\cdot-b=ab$. Moreover, in proving the generic case we will make use of the example that shows $-1\cdot-1=1$. We make a reference *see work above* to indicate the proof has already been shown above.

An Example: $(-1)(-1)=1$

Proof:

$$
\begin{aligned}
1+-1&=0 && \text{AInv}\\
-1(1+-1)&=-1\cdot 0 && \text{CLM}\\
(-1)(1)+(-1)(-1)&=-1\cdot 0 && \text{DL}\\
(-1)(1)+(-1)(-1)&=0 && \text{0MT}\\
-1+(-1)(-1)&=0 && \text{M.Id}\\
1-1+(-1)(-1)&=1+0 && \text{CLA}\\
0+(-1)(-1)&=1+0 && \text{A.Inv}\\
(-1)(-1)&=1 && \text{0 is add identity Voalllaaaa!!}
\end{aligned}
$$

The Theorem [NNT]: $-a\cdot-b=ab$

Proof:

$$\begin{aligned} -a \cdot -b &= -1 \cdot a \cdot -1 \cdot b && \text{MT} \\ &= -1 \cdot -1 \cdot a \cdot b && \text{CoLM} \\ &= (-1 \cdot -1)ab && \text{ALM} \\ &= 1 \cdot ab && \text{see work above} \\ &= ab && \text{M.Id} \end{aligned}$$

Negative Times Positive Theorem[NPT]

In plain words, this theorem says that when a negative times a positive number is always negative. Using math symbols we can make the theorem more precise.

$$(-a)(b) = -(ab)$$

Proof:

$$\begin{aligned} (-a)(b) &= (-1 \cdot a)(b) && \text{MT} \\ &= -1(ab) && \text{ALM} \\ &= -(ab) && \text{MT} \end{aligned}$$

Practice Using These Theorems

(1) Calculate $-3 \cdot -5 \cdot -2$

$$\begin{aligned} -3 \cdot -5 \cdot -2 &= (-3 \cdot -5) \cdot -2 && \text{ALM} \\ &= 15 \cdot -2 && \text{NNT} \\ &= -32 && \text{NPT} \end{aligned}$$

(2) Calculate $-3(5+7)$

$$\begin{aligned} -3(5+7) &= -3 \cdot 12 && \text{AT} \\ &= -36 && \text{NPT} \end{aligned}$$

(3) Calculate $-3(5+-7)$

$$\begin{aligned} -3(5+-7) &= -3\cdot 5 + -3\cdot -7 && \text{DL}\\ &= -15+21 && \text{NPT,NNT}\\ &= -15+(15+6) && \text{AT}\\ &= (-15+15)+6 && \text{ALA}\\ &= 0+6 && \text{AInv}\\ &= 6 && \text{A Id} \end{aligned}$$

(4) Calculate $-2(5^2+-10)$

$$\begin{aligned} -2(5^2+-10) &= -2(5\cdot 5+-10) && \text{N-Expo}\\ &= -2(25+-10) && \text{TT}\\ &= -2(15+10+-10) && \text{AT}\\ &= -2(15+0) && \text{A.Inv}\\ &= -2(15) && \text{AId}\\ &= -30 && \text{NPT} \end{aligned}$$

(5) Calculate $(-3)^2(-5\cdot 2+-7)$

$$\begin{aligned} (-3)^2(-5\cdot 2+-7) &= (-3)(-3)(-5\cdot 2+-7) && \text{N-Exp}\\ &= 9(-10+-7) && \text{NNT,NPT}\\ &= 9(-1\cdot 10+-1\cdot 7) && \text{MT}\\ &= 9\cdot -1(10+7) && \text{DL}\\ &= 9\cdot -1(10+7) && \text{DL}\\ &= 9\cdot -1\cdot 17 && \text{AT}\\ &= 9(-17) && \text{MT}\\ &= -153 && \text{NPT} \end{aligned}$$

(6) Calculate $(-3+5)(4+5)$

$$\begin{aligned} (-3+5)(4+5) &= (-3+5)(9) && \text{AT}\\ &= -3\cdot 9+5\cdot 9 && \text{DL}\\ &= -27+45 && \text{NPT, TT}\\ &= -27+27+18 && \text{AT}\\ &= 0+18 && \text{A.Inv}\\ &= 18 \end{aligned}$$

(7) Calculate $(-3+5)(4+5)$ (second solution...there are many....)

$$\begin{aligned}
(-3+5)(4+5) &= (-3+3+2)(4+5) && \text{AT}\\
&= (0+2)(4+5) && \text{A.Inv}\\
&= (2)(4+5) && \text{A.Id}\\
&= (2)(9) && \text{A.Id}\\
&= 18 && \text{TT}
\end{aligned}$$

Excercises 2.1

For 1-9 Determine if the statement is true or false. If true, provide appropriate justification. For the other exercises, simplify showing every step.

(1) $-3+-5=8$
(2) $5\cdot 0=0$
(3) $-3\cdot -5=15$
(4) $-4\cdot 5=20$
(5) $-5=-1\cdot 5$
(6) $-(5+x)=-1(5+x)$
(7) $-x+-5=-(x+5)$
(8) $-1\cdot x+-1\cdot 5=-1(x+5)$
(9) $-3+-5=-8$
(10) $(-2)(-2)$
(11) -2^2
(12) $(-3)(-3)$
(13) -3^2
(14) $(-2)^3$
(15) $-4^2+(5)(-2)(-3)$
(16) $(5\cdot 3)(2)$
(17) $-(2-3)(4)$
(18) $-(5^2-3^2)$
(19) $(3+5)(6-7)$
(20) $(3+5)(2+5)$
(21) $(-1)^7$
(22) -1^7
(23) $(-1)^8$
(24) -1^8
(25) $(-1)^{76}$
(26) $(-1)^{1997}$
(27) $(-2)^6$
(28) $(6-7)(7-6)(7+6)-1$
(29) $(3-5)^3+(3-1)^5+(3-5)^4$
(30) $9(3+-5)$
(31) $(-2+5)(-3)$
(32) $-9(3+-5)$
(33) $(-2+5)(-3+1)$
(34) $(-2+-5)(-3+1)^2$
(35) $(-2)(-2)$
(36) -2^2
(37) $(-3)(-3)$
(38) -3^2
(39) $(-2)^3$
(40) $-4^2+(5)(-2)(-3)$
(41) $(3+5)(2+5)$
(42) $(-1)^8$
(43) -1^8
(44) $(-1)^7$
(45) $(-2)^6$

2.2. Adding Integers

" .Keep in mind in our quest to construct our understanding of algebra without any preconceived notions, we assume nothing...... "

Gameplan 2.2
(1) *Adding* $\mathbb{N}$
(2) *Adding 0*
(3) *Adding Negatives [N+NT]*
(4) *Mixed*
(5) *Subtracting*

IDEA

Recall, the integers are natural numbers, 0 or negatives. In this section we will learn to add any combination of these. By the end we will know how to add any pair of integers. First we review the adding of two positive integers, then sums involving 0, then sums of negative integers, and finally sums of integers of opposite sign.

ADDING IN $\mathbb{N}$

Note we have already learned to add natural numbers. We did so by adopting the addition table for natural numbers. The following examples should help review the process.

(1) $3 + 8 = 11$ [AT]
(2) $3 + 12 = 15$ [A.T.]
(3) $3 + 10 = 13$ [A.T.]

ADDING 0

Again, you should find this to be review of previously learned axioms, namely, the identity axiom for addition.

(1) $0 + 8 = 8$ [AId]
(2) $3 + 0 = 3$ [AId]
(3) $0 + 0 = 0$ [AId]

NEGATIVE PLUS NEGATIVE THEOREM [N+NT]

Negative Plus Negative Theorem [N+NT] says that if we add two negative numbers, the result is a large negative number. For example, $-2 + -3 = -5$. To prove this, we will first illustrate the argument for these particular number $-2 + -3$, then we will prove it for generic numbers a and b. That is, we will prove that for any integers a and b, we have

$$-a + -b = -(a + b) \qquad \text{[N+NT]}$$

Compute $-2 + -3$

Solution:

$$\begin{aligned} -2 + -3 &= -1 \cdot 2 + -1 \cdot 3 && \text{MT} \\ &= -1(2+3) && \text{DL} \\ &= -1 \cdot 5 && \text{AT} \\ &= -5 && \text{MT} \end{aligned}$$

Now we prove the general [N+NT] theorem, specifically

$$-a + -b = -(a+b)$$

Proof:

$$\begin{aligned} -a + -b &= -1 \cdot a + -1 \cdot b && \text{MT} \\ &= -1(a+b) && \text{DL} \\ &= -(a+b) && \text{MT} \end{aligned}$$

SUBTRACTING

Keep in mind in our quest to construct our understanding of algebra without any preconceived notions, we assume nothing. The concept of *subtracting* is a totally new concept for us. Under this assumption we need to specify exactly what it means to us. In English words, to subtract means;

'subtract' = 'add the additive inverse'

Using math symbols, we can define what subtraction is as:

$$a - b = a + -b \qquad [\text{def } a-b]$$

For example, $3-2$ means $3+-2$. In other words, we can turn every subtraction problem into an addition problem. The beauty in this is that we have already developed many theorems for addition, DL, ALA, CLA, etc... While we don't have any theorems for subtraction. Turning every subtraction problem into an addition problem will prove very helpful. Some examples illustrating our newly found definitions and theorems are in order.

PRACTICE

(1) calculate $3+5$

$$3 + 5 = 8 \qquad \text{A.T}$$

(2) Calculate $8 + -3$

$$\begin{aligned} 8 + -3 &= (5+3) + -3 && \text{A.T} \\ &= 5 + (3 + -3) && \text{ALA} \\ &= 5 + 0 && \text{A.Inv} \\ &= 5 && \text{A.Id} \\ 8 + -3 &= 5 && \text{TP} \end{aligned}$$

(3) Calculate $10 - 2$

$$\begin{aligned} 10 - 2 &= 10 + -2 && \text{Def of } a - b \\ &= (8+2) + -2 && \text{A.T} \\ &= 8 + (2 + -2) && \text{ALA} \\ &= 8 + 0 && \text{A.Inv} \\ &= 8 && \text{A.Id} \\ 10 - 2 &= 8 && \text{T.P.} \end{aligned}$$

(4) Calculate $15 + -2 + -8$

$$\begin{aligned} 15 + -2 + -8 &= (5+10) + -2 + -8 && \text{A.T.} \\ &= (5+2+8) + -2 + -8 && \text{A.T.} \\ &= 5+2+8 + -2 + -8 && \text{ALA} \\ &= 5 + -2 + 2 + 8 + -8 && \text{CoLA} \\ &= 5 + (-2+2) + (8 + -8) && \text{ALA} \\ &= 5 + 0 + 0 && \text{A.Inv} \\ &= 5 && \text{A.Id} \end{aligned}$$

(5) Calculate $15 + -2 + -8$ (same problem as above, alternative solution)

$$\begin{aligned} 15 + -2 + -8 &= 15 + -10 && \text{N+NT} \\ &= (5+10) + -10 && \text{A.T.} \\ &= 5 + (10 + -10) && \text{ALA} \\ &= 5 + 0 && \text{A.Inv} \\ &= 5 && \text{A.Id} \end{aligned}$$

(6) Compute $-2 + -3$

Solution:

$$-2 + -3 = -5 \qquad \text{N+NT}$$

(7) $-12 + 5$

Solution:

$$\begin{aligned} -12 + 5 &= (-7 + -5) + 5 && \text{N+NT} \\ &= -7 + (-5 + 5) && \text{ALA} \\ &= -7 + 0 && \text{AInv} \\ &= -7 && \text{AId} \end{aligned}$$

(8) $-30 + 10$

Solution:

$$\begin{aligned} -30 + 10 &= (-20 + -10) + 10 && \text{N+NT} \\ &= -20 + (-10 + 10) && \text{ALA} \\ &= -20 + 0 && \text{AInv} \\ &= -20 && \text{AId} \end{aligned}$$

Excercises 2.2

(1) Determine if the statement if true or false. If the statement is true, determine why.

(a) $3 + 10 = 13$
(b) $5 + 7 = 12$
(c) $12 + -12 = 0$
(d) $x + -x = 0$
(e) $-4 + -6 = -10$
(f) $-2 = -1 + -1$
(g) $-5 + -8 = -12$
(h) $(c + a) + t = t + (c + a)$
(i) $(c + a) + t = c + (a + t)$
(j) $-(x + 3) + (x + 3) = 0$
(k) if $x + 7 = 0$ then $x = -7$
(l) if $-7 + a = 0$ then $a = 7$
(m) if $-1 + (-1)(-1) = 0$ then $(-1)(-1) = 1$
(n) $x - y = x + -y$
(o) $x - y = y - x$
(p) $x + y = y + x$
(q) $\sqrt{4} + 5 = 5 + \sqrt{4}$
(r) $\sqrt{5} + 0 = \sqrt{5}$

(2) Use only Axioms/Definitions/orTheorems to Calculate the following. Show, justify and understand every step.

(a) $5 + -2$
(b) $10 + 7 + -2$
(c) $-5 + 10 - 2$
(d) $20 - 13$
(e) $100 + -5 + -7$
(f) $-2 + -7$
(g) $-5 + -8$
(h) $-10 + 5$
(i) $-100 + -35 + 10$

2.3. Factoring Integers

" .. until one day, nothing happened,...... "

Gameplan 2.3
(1) *To Factor*
(2) *Prime Numbers*
(3) *Famous Prime Stories*
(4) *Prime Factorization*
(5) *Famous Factoring Stories*
(6) *Famous Factoring Tricks*

To Factor

To factor means to break up into pieces that are multiplied. For example, the number 6 can be 'broken up' and written as $6 = 2 \cdot 3$. These 'pieces' that are multiplied to yield 6 are called factors. We emphasize the fact that these pieces are *multiplied* to yield 6. Consider a different type of 'breaking up' of 6, like $6 = 3 + 3$. This is not a factorization of 6. In fact, to break up a number into the sum of two numbers is 'to *partition* the number', while breaking up a number into the product of two (or more) quantities is 'to factor the number'. Consider the integer 24. $24 = 3 \cdot 8$ is a factorization; $24 = 10 + 14$ is a partition; $24 = 2 \cdot 12$ is another factorization; $24 = 3 \cdot 2 \cdot 4$ is yet another factorization. A natural question arises immediately. How many different factorization of 24 are there? The answer is that there is a bunch of factorization of 24. However there is one very special factorization for every positive integer. This special factorization is called 'prime factorization', and understanding it requires that we understand primes. Before we discuss primes, we summarize what it means to factor.

f a is written as $a = bc$ we say that a is factored and we call b and c factors or divisors of a.

Prime Numbers

A positive integer, p, is called a prime if the only possible factorizations are

$$p = 1 \cdot p \quad \text{or} \quad p = -1 \cdot -p$$

Such factorizations are called trivial factorizations. Famous examples of primes are 1, 2, 3, 5, 7, 11, 13, 17, etc.

The Goldbach Conjecture Story

Prime numbers are/have been the subject of legends and myths. It is said that once a woman sat down and pondered on these primes numbers and pondered and came up with a very mysterious thought. She thought, "hamm... if I take any positive even integer, I can always partition it into two primes..." She went on to partition $2 = 1 + 1$, into two primes. Then she partition $4 = 2 + 2$ into two primes, then $6 = 3 + 3$, then $8 = 3 + 5$, then $10 = 5 + 5$, then $12 = 5 + 7$... and on and on and on... She worked on it all day. Tea time came and she completely forgot about it. Then dinner time came and she completely forgot about it..Night and morning came and it dawned on her she had forgotten to sleep. She realized she could partition any even number she attempted into two primes, BUT she did not know why or if there was some even number out there somewhere that could not be partitioned.

Word got around and soon many mathematicians had hear the problem. Everyone wanted to prove this would always work. Everyone wanted to be the first to explain how it works. Until one day, nothing happened, years went by mathematicians died (and went to heaven) and new ones were born and still, this problem defeated all on them... until one day, nothing happened, until today nothing has happened. As this note is written the problem is still unsolved and carries with it fame, fortune and mathematical immortality to whomever solves it.

Prime Factorization

Now we return to the quest for the most special factorization of an integer, the "prime factorization". To prime factorize means to factor, factor, until you can factor no more. This will happen when all the factors are prime numbers, hence the name prime factorization. There is a very special theorem that says that every positive integer has a unique* prime factorization into positive primes. It may be humbling to know that for really large integers we [the entire human race, the smartest scientist in that have walked this earth] still don't know how to factor them efficiently. Try factoring 178318937164307. Actually, this number is not that large a computer can do it in a few seconds but larger numbers (500 or more digits) are generally very difficult to factor. Even the fastest computers all put together can not handle this simple little factoring task. For us, we will be content to learn and practice 'prime factorizing' of small integers. Some examples are in order.

(1) Prime factorize 24

$$\begin{aligned} 24 &= 6 \cdot 4 && \text{TT} \\ &= 2 \cdot 3 \cdot 2 \cdot 2 && \text{TT} \\ &= 2 \cdot 2 \cdot 2 \cdot 3 && \text{CoLM} \\ &= 2^3 \cdot 3 && \text{+Expo} \end{aligned}$$

(2) Prime factorize 36

$$\begin{aligned} 36 &= 6 \cdot 6 && \text{TT} \\ &= 2 \cdot 3 \cdot 2 \cdot 3 && \text{TT} \\ &= 2 \cdot 2 \cdot 3 \cdot 3 && \text{CoLM} \\ &= 2^3 \cdot 3^2 && \text{+Expo} \end{aligned}$$

(3) Prime factorize 1200

$$\begin{aligned}
1200 &= 120 \cdot 10 && \text{TT}\\
&= 2 \cdot 6 \cdot 2 \cdot 5 && \text{TT}\\
&= 2 \cdot 2 \cdot 3 \cdot 2 \cdot 5 && \text{TT}\\
&= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 && \text{CoLM}\\
&= 2^3 \cdot 3 \cdot 5 && \text{+Expo}
\end{aligned}$$

(4) Prime factorize 1989

$$\begin{aligned}
1989 &= 3 \cdot 663 && \text{TT}\\
&= 3 \cdot 3 \cdot 221 && \text{TT}\\
&= 3 \cdot 3 \cdot 13 \cdot 17 && \text{CoLM}\\
&= 3^2 \cdot 13 \cdot 17 && \text{+Expo}
\end{aligned}$$

The Cryptography Story

The examples above are very easy to factor. They are easy, mainly, because the numbers are relatively small. As mentioned before, for large numbers factoring can be a very serious proposition. Consider factoring the number

$$12312893739287391273984732343 21.$$

In fact, if the numbers are large enough, factoring is so difficult that encryption codes are built around this difficulty. In other words, messages, passwords, account numbers, PIN, or other sensitive information is encrypted using a huge huge number. The coded message is sent along with the huge number. To break the code, it is necessary to factor the number. Because factoring the number is so difficult, very few people can break the code. A whole industry is built around this field. With the growing need of internet security, many people make a living factoring numbers. In fact, one of the leading companies, RSA, keeps track of how well the general public can factor big numbers by offering cash rewards for the factorization of certain numbers. You are invited to look it up on the web, you may find it profitable.

Nice Factoring Tricks

Note the following should be considered hints not theorems. Unfortunately, our journey will fall short of developing all the machinery needed to explain why these statements are true. We will not see a proof of these in this course. The interested student is encouraged to pick up a book on Elementary Number Theory.

(1) Always try what the most obvious first. If a number ends in an even number immediately factor into a 2 times some other $stuff$. i.e. $1998 = 2 \cdot 999$
(2) If a number ends in a '0' immediately factor the 10. i.e. $130 = 10 \cdot 13$

(3) If a number ends in a 5, smile. You can immediately factor out a 5. i.e. $1995 = 5 \cdot 199$

(4) (my favorite trick) The sum of the digits of an integer are divisible by three if and only if the integer is also divisible by three. i.e. the sum of the digits of 62301 is 12, 12 is divisible by 3, therefore 3 should go evenly into 62301. A calculator confirm our beautiful suspicion $62301 = 3 \cdot 20767$

(5) When all else fails, start checking all the primes (up to $\sqrt{theinteger}$) if none of these go evenly into the integer then you can safely assume the integer is prime and can not be factored any more. i.e. to check and see if 113 is a prime we would need to check all the primes less than or equal to $\sqrt{113} = 10.6$. So all we need to check is 2, 3, 5, and 7. Since none of these divide 113 we can conclude 113 is a prime.

(6) Prime factorize 1999. Not even ,does not end in 0 or 5, the sum of the digits is 28 not divisible by 3, there fore we check all primes. $\sqrt{1999} \approx 44.7$. So we only need to check the primes 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, and 43. Since none of these divide 1999, 1999 is a prime number. End of story it is as factored as it going to get.

Excercises 2.3

For 1-9 Determine if the statement is true or false. If true, provide appropriate justification.

(1) $8 = 4 \cdot 2$ is factored
(2) $8 = 4 \cdot 2$ is Prime Factorized
(3) $8 = 2 \cdot 2 \cdot 2$ is Prime Factorized
(4) $8 = 4 + 4$ is factored
(5) $12 = 3 \cdot 2 + 6$ is factored
(6) $12 = 1 \cdot 12$ is factored
(7) $36 = 2^2 \cdot 3^2$is Prime Factorized
(8) $5 = 5$ is Prime Factorized
(9) $10 = 5 + 5$ is factored

Prime Factorize

(10) 35
(11) 36
(12) 48
(13) 128
(14) 256
(15) 1984
(16) 19842
(17) 2345
(18) 2340
(19) 994
(20) 157
(21) 44
(22) 89
(23) 163
(24) 805
(25) 94
(26) 52
(27) 8
(28) 5
(29) 15
(30) 378221
(31) 12317
(32) 28891

2.4. GCD for Integers

" .. until one day, nothing happened,...... "

Gameplan 2.4
(1) *Informal GCD*
(2) *GCD by silliest method*
(3) *GCD by Inspection*
(4) *GCD the professional way*

Informal Definition of GCD

We can offer two different, equivalent definitions for the *greatest common divisor.* The first is an informal definition, while the second is a bit more precise.

The informal definition is exactly what the name says, 'the greatest common factor'. Suppose we wanted to find the gcd(18,24). The informal definition says we should find all divisors, then all common divisors, then pick the greatest common divisor.

First we find all divisors for each number:
divisor for 18 : $1, 2, 3, 6, 18$
divisors for 24 : $1, 2, 3, 4, 6, 12, 24$
Then we identify the common divisors: $1, 2, 3, 6$
Then we pick the greatest of the common divisors, 6, and we conclude

$$gcd(18, 24) = 6$$

This is method is not very sophisticated, for lack of a better name we call it 'the silliest method'

GCD by Inspection

For relatively small numbers, we can carry out the above calculations without writing much. In other words, we could just sit, think for a second and realize upon inspection $gcd(18, 24) = 6$. A couple other examples we could do by inspection, [BI]:

(1) $gcd(6, 15) = 3$ [BI]
(2) $gcd(16, 15) = 1$ [BI]
(3) $gcd(16, 80) = 16$ [BI]
(4) $gcd(25, 15) = 5$ [BI]
(5) $gcd(30, 80) = 10$ [BI]

GCD, the Professional Way

Here we present the second definition of GCD. This definition will give us a third option for finding gcd's, and it prove useful in future chapters when we find gcd's for polynomials. Feel free to read it a couple of times until it makes sense.

Suppose a and b are a positive numbers. Given their respective prime factorizations, we define $gcd(a, b)$ as the product of common primes, each prime raised to the lower of the two powers.

(1) Find gcd(4,8)

$$\begin{aligned} 4 &= 2 \cdot 2 && \text{TT} \\ 4 &= 2^2 && \text{+Expo} \\ 8 &= 2 \cdot 2 \cdot 2 && \text{TT} \\ 8 &= 2^3 && \text{+Expo} \\ gcd(4, 8) &= 2^2 && \text{GCD def} \end{aligned}$$

(2) Find gcd(320,84)

$$\begin{aligned} 320 &= 2^3 \cdot 3^2 \cdot 5 && \text{TT, +Expo} \\ 84 &= 2^2 \cdot 3 \cdot 7 && \text{TT, +Exp} \\ gcd(320, 84) &= 2^{\square}3^{\square} && \text{these are the only common primes} \\ gcd(320, 84) &= 2^2 3^1 && \text{GCD def} \end{aligned}$$

(3) Find gcd(320,252)

$$\begin{aligned} 320 &= 2^3 \cdot 3^2 \cdot 5 && \text{TT, +Exp} \\ 252 &= 2^2 \cdot 3^2 \cdot 7 && \text{TT, +Exp} \\ gcd(320, 84) &= 2^{\square}3^{\square} && \text{these are the only common primes} \\ gcd(320, 84) &= 2^2 3^2 && \text{GCD def} \end{aligned}$$

(4) Find $gcd(x^3y^5z, x^6y^2w)$

$$\begin{aligned} x^3y^5z &= x^3y^5z && \text{RP} \\ x^6y^2w &= x^6y^2w && \text{RP} \\ gcd(x^3y^5z, x^6y^2w) &= x^{\square}y^{\square} && \text{these are the only common primes} \\ gcd(x^3y^5z, x^6y^2w) &= x^3y^2 && \text{GCD def} \end{aligned}$$

(5) Find $gcd(x^3y^5z^3, x^6y^2wz^{10})$

$$\begin{aligned} x^3y^5z^3 &= x^3y^5z^3 && \text{RP} \\ x^6y^2wz^{10} &= x^6y^2wz^{10} && \text{RP} \\ gcd(x^3y^5z^3, x^6y^2wz^{10}) &= x^{\square}y^{\square}z^{\square} && \text{these are the only common primes} \\ gcd(x^3y^5z^3, x^6y^2wz^{10}) &= x^3y^2z^3 && \text{GCD def} \end{aligned}$$

Excercises 2.4
Find each Greatest Common Divisor

(1) $\gcd(1, 100)$
(2) $\gcd(5, 30)$
(3) $\gcd(24, 36)$
(4) $\gcd(18, 28)$
(5) $\gcd(0, 200)$
(6) $\gcd(4, 12)$
(7) $\gcd(6, 37)$
(8) $\gcd(9, 33)$
(9) $\gcd(x^3y^2, z^5x^3y)$
(10) $\gcd(20, 30, 55)$
(11) $\gcd(12, 28, 18)$
(12) $\gcd(10, 18)$
(13) $\gcd(6x^5, 10x^2y)$
(14) $\gcd(24, 48)$
(15) $\gcd(100, 101)$
(16) $\gcd(200, 201)$
(17) $\gcd(300, 301)$
(18) $\gcd(96, 240)$
(19) $\gcd(1998, 2560)$
(20) $\gcd(333, 4444)$
(21) $\gcd(19, 43)$
(22) $\gcd(2^3 3^7, 2^5 3^4)$
(23) $\gcd(7^6 3^7, 7^2 3^{10})$
(24) $\gcd(1997, 2 \cdot 1997)$
(25) Suppose the US government made only coins worth 34 cents and worth 85 cents. How would you buy a 1 cent pencil?

2.5. LCM for Integers

" .. until one day, nothing happened,...... "

Gameplan 2.5
(1) *Informal LCM*
(2) *LCM by silliest method*
(3) *LCM by Inspection*
(4) *LCM the professional way*

Informal LCM

The least common multiple is the gcd's close relative. It is commonly used to add fractions together when they have a different denominator. We get an intuitive understanding of it from its name. It is the *least, common, multiple.* Suppose we wanted to find the LCM(6,8). All we need is a list of all positive multiples of 6:

$$\text{Multiples of } 6 : 6, 12, 18, 24, 30, 36, 42, 48, 54, ...$$

Then we list all positive multiples of 8:

$$\text{Multiples of } 8 : 8, 16, 24, 32, 40, 48, 56, ...$$

We identify the common multiples:

$$\text{Common Multiples of 6 and } 8 : 24, 48, 72, ...$$

Then we simply pick the *Least* common multiple, namely 24;

$$LCM(6, 8) = 24$$

LCM by Inspection

After some practice, you may feel very comfortable carrying out the above calculations without actually listing any numbers. It is true especially of small numbers, that their LCM can be calculated by inspection. If this is the case, we may justify the finding of an LCM *'by inspection [BI]'*. See the examples below.

(1) Example: find $lcm(2, 3)$

Solution:

$\text{lcm}(2, 3) = 6$ [BI]

(2) Example: find $lcm(5, 10)$

Solution:

$\text{lcm}(5, 10) = 10$ [BI]

(3) Example: find $lcm(8, 10)$

Solution:

$\text{lcm}(8, 10) = 40$ [BI]

LCM THE PROFESSIONAL WAY

If a and b are positive , and these are prime factorized, $LCM[a, b]$ is equal to the product of all primes on on either factorization raised to the higher of the two powers for each prime.

EXAMPLES

(1) $LCM[8, 12]$

$$8 = 2^3 \qquad \text{TT, Def of exp}$$
$$12 = 2^2 \cdot 3 \qquad \text{TT, Def of exp}$$
$$LCM[8, 12] = 2^3 \cdot 3 \qquad \text{Def. of LCM}$$

(2) $LCM[32, 12]$

$$32 = 2^5 \qquad \text{TT, Def of exp}$$
$$12 = 2^2 \cdot 3 \qquad \text{TT, Def of exp}$$
$$LCM[32, 12] = 2^5 \cdot 3 \qquad \text{Def. of LCM}$$

(3) $LCM[32, 24]$

$$32 = 2^5 \qquad \text{TT, Def of exp}$$
$$24 = 2^3 \cdot 3 \qquad \text{TT, Def of exp}$$
$$LCM[32, 24] = 2^5 \cdot 3 \qquad \text{Def. of LCM}$$

(4) Find $gcd(x^3y^5z^3, x^6y^2wz^{10})$

$$x^3y^5z^3 = x^3y^5z^3 \qquad \text{RP}$$
$$x^6y^2wz^{10} = x^6y^2wz^{10} \qquad \text{RP}$$
$$lcm(x^3y^5z^3, x^6y^2wz^{10}) = x^6y^5z^{10}w \qquad \text{LCM def}$$

Excercises 2.5

(1) $\text{lcm}(1, 100)$
(2) $\text{lcm}(5, 30)$
(3) $\text{lcm}(24, 36)$
(4) $\text{lcm}(18, 28)$
(5) $\text{lcm}(0, 200)$
(6) $\text{lcm}(4, 12)$
(7) $\text{lcm}(6, 37)$
(8) $\text{lcm}(9, 33)$
(9) $\text{lcm}(15, 100)$
(10) $\text{lcm}(20, 30, 55)$
(11) $\text{lcm}(12, 28, 18)$
(12) $\text{lcm}(10, 18)$

(13) $\operatorname{lcm}(18, 42)$
(14) $\operatorname{lcm}(24, 48)$
(15) $\operatorname{lcm}(100, 101)$
(16) $\operatorname{lcm}(200, 201)$
(17) $\operatorname{lcm}(300, 301)$
(18) $\operatorname{lcm}(96, 240)$
(19) $\operatorname{lcm}(1998, 2560)$
(20) $\operatorname{lcm}(333, 4444)$
(21) $\operatorname{lcm}(19, 43)$
(22) $\operatorname{lcm}(2^3 3^7, 2^5 3^4)$
(23) $\operatorname{lcm}(7^6 3^7, 7^2 3^{10})$
(24) $\operatorname{lcm}(1997, 2 \cdot 1997)$
(25) $\operatorname{lcm}[x^2 y, x^3 z^2]$

2.6. Clock Arithmetic

" .. until one day, nothing happened,...... "

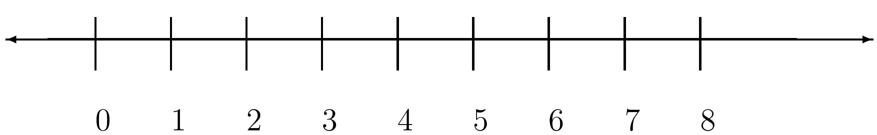
Gameplan 2.6

(1) *Imagine* $\mathbb{Z}_7$
(2) *Addition Table in* $\mathbb{Z}_7$
(3) *Multiplication Table in* $\mathbb{Z}_7$
(4) *Solve equations in* $\mathbb{Z}_7$

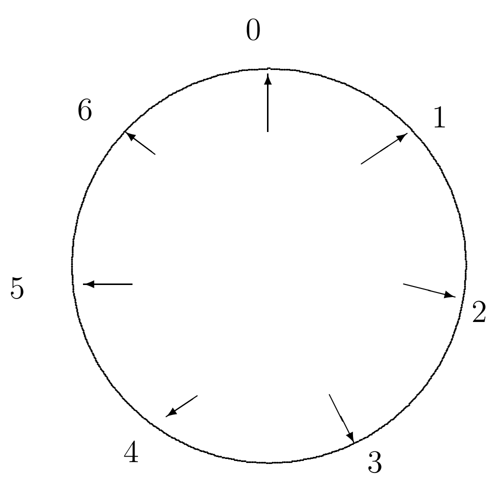
IMAGINE

Imagine a world of numbers much simpler than ours. Imagine a worlds with *only* 7 numbers. We would count: *zero, one, two, tree, four, five, six, zero, one two, three, four, five, six, zero, one,...*

Usually, we use the number line to picture our numbers...



In some sense this is a 'linear' way of counting. The proposal here is to count circularly, rather than linearly. Thus, instead of having a number line, imagine having a number circle



This set of seven numbers is called *the integers modulo 7*, $\mathbb{Z}_7$. There is nothing sacred about 7. We could have just as easily considered a world with 5 integers or 10 or 12, or any number n. In any case, we now take a moment to play with the idea of adding and subtracting in this seven-number world. In addition, we will venture to ask, which of the axioms we've learned for regular numbers hold true here.

Addition/Multiplication Tables

Consider what we get if we sum $4+5$. On a regular day we'd get 9. However, in this seven-number world, there is no 9. The way we will add $4+5$ is to simply count 9 units around the 7-number circle starting at 0, and see where we end up. We would end up 2. Under this definition of adding in this 7-number world we have

$$4+5=2$$

A slight drawback here is that this way of adding seems to contradict the previous axiom *addition table for naturals.* To avoid confusion we must figure out a way to distinguish the 7-number world from the regular number. We do this by changing to '=' sign to a congruence sign $\equiv$. This way there is no confusion, whenever we say 'equal' we will know we are in the regular number world, and whenever we say 'congruent' we will know we are in the world modulo 7, $\mathbb{Z}_7$.

So we prefer to write

$$4+5\equiv 2.$$

We can easily complete an addition table for $\mathbb{Z}_7$. We complete a few more examples and leave it as an important exercise for the reader to finish the addition table.

(1) $4+3$

Solution:

On a regular day, $4+3=7$, but there is no 7 in $\mathbb{Z}_7$, so we start counting around the circle. We count 4 and then we count 3 more and end up at 0, therefore

$$4+3\equiv 0$$

(2) $0+-5$

Solution:

On a regular day, $0+-5=-5$, but there is no -5 in $\mathbb{Z}_7$, so we start counting around the circle. To count negative numbers, we count around the circle *counterclockwise.* So we start at 0, count counterclockwise, $-1,-2,-3,-4,-5$ and end at 2. We summarize

$$0+-5\equiv 2$$

(3) $5\cdot 3$

Solution:

On a regular day, $5\cdot 3=15$, but there is no 15 in $\mathbb{Z}_7$, so we start counting around the circle. We count around the circle starting at 0, 1,2, 3,..., 15, until we end at 1. We summarize

$$5 \cdot 3 \equiv 1$$

This example, gives us further insight into inverses. Recall when two numbers kill each other to give us 1, they are by definition inverses. Since $5 \cdot 3 \equiv 1$, we know 5 and 3 are multiplicative inverses in the $\mathbb{Z}_7$ world!

SOLVE AN EQUATION

Now we gather our courage and go where few dare. We contemplate the thought of solving an equation *modulo 7.* Solve

$$3x + 2 \equiv 6$$

Solution:

$3x + 2 \equiv 6$	Given
$3x + 2 + -2 \equiv 6 + -2$	CLA
$3x + 0 \equiv 6 + -2$	AInv
$3x \equiv 6 + -2$	AId
$3x \equiv 4 + 2 + -2$	AT for $\mathbb{Z}_7$
$3x \equiv 4 + 0$	Ainv
$3x \equiv 4$	Aid
$5 \cdot 3x \equiv 5 \cdot 4$	CLM
$1 \cdot x \equiv 6$	TT for $\mathbb{Z}_7$
$x \equiv 6$	Mid

This example is intended to help solidify and deepen your understanding of the axioms, and the beauty of reasoning. The next example, is intended to tickle your brain and explore the possibilities. Consider the following. In the regular number world, there are infinite many numbers, while in $\mathbb{Z}_7$, there are only 7 possible numbers. Now, consider solving the equation

$$2x^3 + 3x^2 + 1 \equiv 5$$

In the regular world, solving this may be way out of our league. In the modulo 7 world, we have a new and fresh possibility, namely, to exhaust all 7 possible solutions. In other words, if the equation has a solution, it must be one (or more) of the seven numbers so we can just try each one, one by one until we find a a solution. Notice this is impossible for regular numbers because there are infinite many of them.

We begin by checking '1' to see if it solves:

$$\begin{aligned}
2x^3 + 3x^2 + 1 &\equiv && \text{given}\\
&\equiv 2 \cdot 1^3 + 3 \cdot 1^2 + 1 && \text{Sub } x = 1\\
&\equiv 2 \cdot 1 + 3 \cdot 1 + 1 && \text{TT for } \mathbb{Z}_7\\
&\equiv 2 + 3 + 1 && \text{TT for } \mathbb{Z}_7\\
&\equiv 6 && \text{AT for } \mathbb{Z}_7\\
&\neq 5
\end{aligned}$$

We go on and check $x = 2$,

$$\begin{aligned}
2x^3 + 3x^2 + 1 &\equiv && \text{given}\\
&\equiv 2 \cdot 2^3 + 3 \cdot 2^2 + 1 && \text{Sub } x = 2\\
&\equiv 2 \cdot 1 + 3 \cdot 4 + 1 && \text{TT for } \mathbb{Z}_7\\
&\equiv 2 + 5 + 1 && \text{TT for } \mathbb{Z}_7\\
&\equiv 0 && \text{AT for } \mathbb{Z}_7\\
&\neq 5
\end{aligned}$$

Then we check $x = 3$

$$\begin{aligned}
2x^3 + 3x^2 + 1 &\equiv && \text{given}\\
&\equiv 2 \cdot 3^3 + 3 \cdot 3^2 + 1 && \text{Sub } x = 3\\
&\equiv 2 \cdot 6 + 3 \cdot 2 + 1 && \text{TT for } \mathbb{Z}_7\\
&\equiv 5 + 6 + 1 && \text{TT for } \mathbb{Z}_7\\
&\equiv 5 && \text{AT for } \mathbb{Z}_7
\end{aligned}$$

We have found a solution $x = 3$ solves the equation!

Excercises 2.6

Compute all in $\mathbb{Z}_7$

(1) $3 + 5$
(2) 4^2
(3) $3 \cdot 4 + 5$
(4) $5 + -7$
(5) Complete the Addition Table for $\mathbb{Z}_7$

+	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

(6) Complete the Multiplication Table for $\mathbb{Z}_7$

×	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

(7) Solve $3x + 2 \equiv 1$
(8) Solve $3x^2 + 2 \equiv 1$
(9) what is the multiplicative inverse of 4?
(10) what is the multiplicative inverse of 2?

CHAPTER 3

The Rationales

3.1. Multiplying in $\mathbb{Q}$

" .. until one day, nothing happened,...... "

Gameplan 3.1

(1) *What are* $\mathbb{Q}$
(2) *JOT*
(3) *OUT*
(4) *MBT*
(5) *MAT*
(6) *NWW*

RATIONAL NUMBERS DEFINED

Our discussions thus far have involved very few fractions. In fact, we have only seen fractions of the type $\frac{1}{3}$, or $\frac{1}{5}$, in general for any nonzero a, we have seen and defined $\frac{1}{a}$ as *the multiplicative inverse of* a. In other words, it is the number that, when multiplied by a, yields 1. Some examples illustrating this definition should help refresh the concept.

(1) $3 \cdot \frac{1}{3} = 1$ [Minv]
(2) $5 \cdot \frac{1}{5} = 1$ [Minv]
(3) $7 \cdot \frac{1}{7} = 1$ [Minv]

Now, that we know and remember exactly what a fraction, $\frac{1}{5}$, is, we are ready to construct a good definition for a general fraction of the type, $\frac{3}{5}$.

We will define $\frac{3}{5}$ as just a short way of writing $3 \cdot \frac{1}{5}$. Said another way, the definition of the general fraction $\frac{a}{b}$ is

$$\frac{a}{b} = a \cdot \frac{1}{b} \qquad \text{by def } \frac{a}{b}$$

Below we illustrate the definition of fractions via a couple of examples.

(1) $\frac{4}{7} = 4\frac{1}{7}$ [Def $\frac{a}{b}$]
(2) $\frac{23}{10} = 23\frac{1}{10}$ [Def $\frac{a}{b}$]
(3) $\frac{44}{17} = 44\frac{1}{17}$ [Def $\frac{a}{b}$]
(4) $\frac{7}{7} = 7\frac{1}{7}$ [Def $\frac{a}{b}$]
(5) $\frac{1}{1} = 1\frac{1}{1}$ [Def $\frac{a}{b}$]

ANY NUMBER OVER ITSELF IS JUST ONE

Here we consider a special type of fractions. Those with equal numerator and denominator, as $\frac{2}{2}, \frac{5}{5}, \frac{23}{23}, \ldots$ It turns out that any fraction like these is equal to one. We prove it below using definitions and axioms. Once proven we will use this theorem as convenient and refer to it as the *Just One Theorem [JOT].*

(1) Example: $\frac{5}{5} = 1$

$$\begin{aligned} \frac{5}{5} &= 5 \cdot \frac{1}{5} && \text{Def of } \frac{a}{b} \\ &= 1 && \text{M.Inv.} \end{aligned}$$

(2) Example: $\frac{3}{3} = 1$

$$\begin{aligned} \frac{3}{3} &= 3 \cdot \frac{1}{3} && \text{Def of } \frac{a}{b} \\ &= 1 && \text{M.Inv.} \end{aligned}$$

(3) Proof of [JOT]: $\frac{c}{c} = 1$ if $c \neq 0$

$$\begin{aligned} \frac{c}{c} &= c \cdot \frac{1}{c} && \text{Def of } \frac{a}{b}, \\ &= 1 && \text{M.Inv.} \end{aligned}$$

Any number over one is itself

Fractions with a denominator equal to 1 also have a very special feature. Namely, that they are equivalent to the numerator. Below we offer a short proof following a couple of examples. In the future, we will refer to this theorem as the *One Under Theorem [OUT].*

(1) Example: Show $\frac{3}{1} = 3$

$$\begin{aligned} \frac{3}{1} &= 3 \cdot \frac{1}{1} && \text{Def } \frac{a}{b} \\ &= 3 \cdot 1 && \text{JOT} \\ &= 3 && \text{MId} \\ \frac{3}{1} &= 3 && \text{TP} \end{aligned}$$

(2) Example: Show $\frac{7}{1} = 7$

$$\begin{aligned} \frac{7}{1} &= 7 \cdot \frac{1}{1} && \text{Def } \frac{a}{b} \\ &= 7 \cdot 1 && \text{JOT} \\ &= 7 && \text{MId} \\ \frac{7}{1} &= 7 && \text{TP} \end{aligned}$$

(3) Example: Show $\frac{c}{1} = c$

$$\begin{aligned} \frac{c}{1} &= c \cdot \frac{1}{1} && \text{Def } \frac{a}{b} \\ &= c \cdot 1 && \text{JOT} \\ &= c && \text{MId} \\ \frac{c}{1} &= c && \text{TP} \end{aligned}$$

Multiplying the Bottoms Theorem [MBT]

We now consider multiplying fractions with numerator 1. Once we figure out how to multiply fractions with numerator 1, we will be ready to multiply fractions with any numerator or denominator. We will first multiply $\frac{1}{3} \cdot \frac{1}{5}$ to get $\frac{1}{15}$. The same steps can be used to show that in general, to multiply two fractions of the type $\frac{1}{a} \cdot \frac{1}{b}$, we simply multiply the bottoms. This theorem will be called precisely this, *the multiply the bottoms theorem, [MBT].* We now prove $\frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$.

$$\begin{aligned} 3 \cdot 5 \cdot \frac{1}{3 \cdot 5} &= 1 && \text{Minv} \\ \frac{1}{5} \cdot \frac{1}{3} \cdot 3 \cdot 5 \cdot \frac{1}{3 \cdot 5} &= \frac{1}{5} \cdot \frac{1}{3} \cdot 1 && \text{CLM} \\ \frac{1}{5} \cdot 5 \cdot \frac{1}{3} \cdot 3 \cdot \frac{1}{3 \cdot 5} &= \frac{1}{5} \cdot \frac{1}{3} \cdot 1 && \text{CoLM} \\ \left(\frac{1}{5} \cdot 5\right) \left(\frac{1}{3} \cdot 3\right) \frac{1}{3 \cdot 5} &= \frac{1}{5} \cdot \frac{1}{3} \cdot 1 && \text{ALM} \\ 1 \cdot 1 \cdot \frac{1}{3 \cdot 5} &= \frac{1}{5} \cdot \frac{1}{3} \cdot 1 && \text{Minv} \\ \frac{1}{3 \cdot 5} &= \frac{1}{5} \cdot \frac{1}{3} && \text{Mid} \end{aligned}$$

Now the proof for general fractions, [MBT],

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$$

Proof:

$$
\begin{aligned}
ab \cdot \frac{1}{ab} &= 1 && \text{Minv}\\
\frac{\mathbf{1}}{\mathbf{b}} \cdot \frac{\mathbf{1}}{\mathbf{a}} \cdot ab \cdot \frac{1}{ab} &= \frac{\mathbf{1}}{\mathbf{b}} \cdot \frac{\mathbf{1}}{\mathbf{a}} \cdot 1 && \text{CLM}\\
\frac{1}{b} \cdot b \cdot \frac{1}{a} \cdot a \cdot \frac{1}{ab} &= \frac{1}{b} \cdot \frac{1}{a} \cdot 1 && \text{colm}\\
\left(\frac{1}{b} \cdot b\right)\left(\frac{1}{a} \cdot a\right)\frac{1}{ab} &= \frac{1}{b} \cdot \frac{1}{a} \cdot 1 && \text{alm}\\
1 \cdot 1 \cdot \frac{1}{ab} &= \frac{1}{b} \cdot \frac{1}{a} \cdot 1 && \text{minv}\\
\frac{1}{ab} &= \frac{1}{b} \cdot \frac{1}{a} && \text{mid}\\
\frac{1}{ab} &= \frac{1}{a} \cdot \frac{1}{b} && \text{colm}\\
\frac{1}{a} \cdot \frac{1}{b} &= \frac{1}{ab} && \text{sp}
\end{aligned}
$$

Multiply Across Theorems [MAT]

We are now ready for the multiplication of general fractions. As usual, we will begin by showing all the steps for specific numbers, just to show an example of the proof. We will then prove the general case. From here on, we will refer to this as the *Multiply Across Theorem [MAT].*

The Example

$$
\begin{aligned}
\frac{3}{5} \cdot \frac{7}{11} &= 3 \cdot \frac{1}{5} \cdot 7 \cdot \frac{1}{11} && \text{Def of } \frac{a}{b}\\
&= 3 \cdot 7 \cdot \frac{1}{5} \cdot \frac{1}{11} && \text{colm}\\
&= (3 \cdot 7) \cdot \left(\frac{1}{5} \cdot \frac{1}{11}\right) && \text{ALM}\\
&= 21 \cdot \frac{1}{55} && \text{TT,MBT}\\
&= \frac{21}{55} && \text{def of } \frac{a}{b}
\end{aligned}
$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Proof:

$$\begin{aligned}
\frac{a}{b}\cdot\frac{c}{d} &= a\cdot\frac{1}{b}\cdot c\cdot\frac{1}{d} && \text{Def of } \frac{a}{b}\\
&= a\cdot c\cdot\frac{1}{b}\cdot\frac{1}{d} && \text{colm}\\
&= ac\cdot\frac{1}{bd} && \text{MBT}\\
&= \frac{ac}{bd} && \text{Def of } \frac{a}{b}
\end{aligned}$$

Note that the theorem says you can take the product of two fractions and glue them together into one fraction by just multiplying across. Or said another way, it says that you can take one fraction, with a product at the top and a product at the bottom and split the fraction into the product of two fractions. Some examples are in order.

(1) Multiply $\frac{2}{3}\cdot\frac{5}{7}$

Solution:

$$\begin{aligned}
\frac{2}{3}\cdot\frac{5}{7} &= \frac{2\cdot 5}{3\cdot 7} && \text{MAT}\\
&= \frac{10}{21} && \text{TT}
\end{aligned}$$

(2) Multiply $\frac{5}{4}\cdot\frac{2}{7}$

Solution:

$$\begin{aligned}
\frac{5}{4}\cdot\frac{2}{7} &= \frac{5\cdot 2}{4\cdot 7} && \text{MAT}\\
&= \frac{10}{28} && \text{TT}
\end{aligned}$$

NWW Theorem

This theorem tell us exactly how we can deal with negative numbers on fractions. It is a well known fact that the negative sign can go on the top (numerator), on the denominator, or in the middle. That is, the following three fractions are all equivalent.

$$\frac{-1}{8} = \frac{1}{-8} = -\frac{1}{8}$$

It may be a fun and instructive exercise to say the above line using English words for the definitions of each quantity. For example, the first equation says that $\frac{-1}{8} = \frac{1}{-8}$. Using English words and definitions, it says that the 'additive inverse of 1 times the multiplicative inverse of 8 is equal to the multiplicative inverse of the additive inverse of 8.' In any case, we take a moment here to prove the first equality. $\frac{-1}{8} = \frac{1}{-8}$, leaving the second equality, $\frac{1}{-8} = -\frac{1}{8}$, as an exercise. In the future, we will freely move the negative sign on he numerator, denominator or middle, and justify it by this theorem, the *Negative Wherever we Want Theorem [NWW].*

$$
\begin{aligned}
1 + -1 &= 0 && \text{A.Inv.}\\
(1 + -1)\frac{1}{8} &= 0 \cdot \frac{1}{8} && \text{C.L.M.}\\
1 \cdot \frac{1}{8} + -1 \cdot \frac{1}{8} &= 0 \cdot \frac{1}{8} && \text{D.L.}\\
\frac{1}{8} + \frac{-1}{8} &= 0 && \text{MId, Def } \frac{a}{b}\text{, 0MT}\\
-\frac{\mathbf{1}}{\mathbf{8}} + \frac{1}{8} + \frac{-1}{8} &= -\frac{\mathbf{1}}{\mathbf{8}} + 0 && \text{CLA}\\
0 + \frac{-1}{8} &= -\frac{\mathbf{1}}{\mathbf{8}} + 0 && \text{Ainv}\\
\frac{-1}{8} &= -\frac{\mathbf{1}}{\mathbf{8}} && \text{Aid}
\end{aligned}
$$

PRACTICE

(1) Multiply $\frac{3}{5} \cdot \frac{7}{2}$

$$
\begin{aligned}
\frac{3}{5} \cdot \frac{7}{2} &= \frac{3 \cdot 7}{5 \cdot 2} && \text{mat}\\
&= \frac{21}{10} && \text{tt}
\end{aligned}
$$

(2) Multiply $\frac{3}{5} \cdot \frac{7}{2}$

$$
\begin{aligned}
\frac{-3}{5} \cdot \frac{7}{9} &= \frac{-3 \cdot 7}{5 \cdot 9} && \text{mat}\\
&= \frac{-21}{45} && \text{NPT, TT}
\end{aligned}
$$

(3) Multiply $5 \cdot \frac{7}{2}$

$$\begin{aligned} 5 \cdot \frac{7}{9} &= \frac{5}{1} \cdot \frac{7}{9} && \text{jot} \\ &= \frac{5 \cdot 7}{1 \cdot 9} && \text{mat} \\ &= \frac{35}{9} && \text{Mid, TT} \end{aligned}$$

Excercises 3.1

Multiply. Assume any variable and/or denominator represents some non-zero rational number.

(1) $\frac{-2}{4} \cdot \frac{2}{3}$
(2) $\frac{4}{-4} \cdot \frac{1}{-1}$
(3) $\frac{2}{3} \cdot \frac{6}{6}$
(4) $\frac{-2}{-3} \cdot \frac{3}{4}$
(5) $\frac{2}{6} \cdot \frac{7}{7}$
(6) $\frac{8}{7} \cdot \frac{blah}{14}$
(7) $\frac{-3}{-9} \cdot \frac{3}{2}$
(8) $\frac{4}{5} \cdot \frac{10}{30}$
(9) $\frac{5}{7} \cdot \frac{4}{4}$
(10) $\frac{3 \cdot 5}{4 \cdot 5}$
(11) $\frac{4}{5} \cdot \frac{10}{30}$
(12) $(\frac{-2}{3})^2$
(13) $(\frac{5}{12})(-\frac{8}{15})(\frac{1}{3})$
(14) $(\frac{6}{11}) \cdot (\frac{-2}{3})$
(15) $\frac{2}{3} \cdot \frac{7}{4}$
(16) $(\frac{-5}{7})^2(\frac{14}{75})$
(17) $(\frac{-1}{2})^3(-8)$
(18) $(\frac{(-1)^7}{5})(\frac{15}{12})$
(19) $(-\frac{3}{4})(-\frac{8}{27})$
(20) $\frac{3}{5} \cdot 6$
(21) $6 \cdot \frac{3}{5}$

3.2. Simplifying in $\mathbb{Q}$

" .. until one day, nothing happened,...... "

Gameplan 3.2
(1) *What is 'simplified'*
(2) *How to simplify*
(3) *When to simplify*

WHAT IS SIMPLIFIED

We will say that an integer fraction $\frac{a}{b}$ is simplified if and only if $gcd(a, b) = 1$. We will use JOT and MAT to simplify fractions. There is one very important comment to make here with regards to the Multiply Across Theorem [MAT]. Recall MAT says when we multiply two fractions together, we glue them into one fraction, multiply across the top, and across the bottom. As in;

$$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7}$$

Consider what MAT says if we read it from right to left. Reading it from right to left, it says that if we have a fraction with two things multiplied on the top and two things multiplied on the bottom, then we can *unglue* them into the product or two fractions. That it, if we read MAT backwards, the above example can be re-interpreted as

$$\frac{2 \cdot 5}{3 \cdot 7} = \frac{2}{3} \cdot \frac{5}{7}$$

Both of these statements are true by MAT. To simplify fractions, we will make extensive use *unglueing* interpretation of MAT.

PRACTICE USING MAT AND JOT TO SIMPLIFY FRACTIONS

(1) $\frac{5}{3}$

Solution:

Since gcd(5,3)=1, this fraction is already simplified.

(2) $\frac{2}{6}$

Solution:

$$\begin{aligned} \frac{2}{6} &= \frac{2\cdot 1}{2\cdot 3} && \text{TT} \\ &= \frac{2}{2}\cdot\frac{1}{3} && \text{mat (unglue)} \\ &= 1\cdot\frac{1}{3} && \text{jot} \\ &= \frac{1}{3} && \text{mdi} \end{aligned}$$

(3) $\frac{14}{49}$

Solution:

$$\begin{aligned} \frac{14}{49} &= \frac{7\cdot 2}{7\cdot 7} && \text{TT} \\ &= \frac{7}{7}\cdot\frac{2}{7} && \text{mat} \\ &= 1\cdot\frac{2}{7} && \text{jot} \\ &= \frac{2}{7} && \text{mid} \end{aligned}$$

(4) Simplify $\frac{24}{40}$

$$\begin{aligned} \frac{24}{40} &= \frac{8\cdot 3}{8\cdot 5} && \text{TT} \\ &= \frac{8}{8}\cdot\frac{3}{5} && \text{MAT} \\ &= 1\cdot\frac{3}{5} && \text{JOT} \\ &= \frac{3}{5} && \text{MId} \end{aligned}$$

(5) $\frac{4}{6}$

$$\begin{aligned}
\frac{4}{6} &= \frac{2\cdot 2}{3\cdot 2} && \text{TT}\\
&= \frac{2}{2}\cdot\frac{3}{2} && \text{MAT}\\
&= 1\cdot\frac{3}{2} && \text{JOT}\\
&= \frac{3}{2} && \text{MId}
\end{aligned}$$

(6) $\frac{-4x^2}{-6x^3}$

$$\begin{aligned}
\frac{-4x^2}{-6x^3} &= \frac{2\cdot -2x^2}{-6x\cdot x^2} && \text{NPT, JAE}\\
&= \frac{2\cdot -2x^2}{3\cdot -2\cdot x\cdot x^2} && \text{NPT}\\
&= \frac{2\cdot -2x^2}{3x\cdot -2x^2} && \text{CoLM}\\
&= \frac{2}{3x}\cdot\frac{-2x^2}{-2x^2} && \text{MAT}\\
&= \frac{2}{3x}\cdot 1 && \text{JOT}\\
&= \frac{2}{3x} && \text{MId}
\end{aligned}$$

When To Simplify

It is customary to simplify whenever one can. On the other hand, there may be times when simplifying makes matters worse, while not simplifying does not cause any harm. There is room here for you to develop your own math personality, simplify whenever you feel like it, or whenever you see it helpful in solving a particular problem. That being said, you must *know* how to simplify. The following exercises should help hone your simplifying skills.

Excercises 3.2

(1) $\frac{24}{-28}$
(2) $\frac{135}{-30}$
(3) $\frac{-4}{16}$
(4) $\frac{160}{120}$
(5) $\frac{30x^5}{75x^3}$
(6) $\frac{-32}{-4}$
(7) $\frac{-60}{-12}$
(8) $\frac{6x^3}{15x^5}$
(9) $\frac{-28}{16}$
(10) $\frac{-18}{30}$
(11) $\frac{-63}{45}$
(12) $\frac{12x^10}{20x^5}$

(13) $\frac{30}{-75}$
(14) $\frac{4}{-4}$
(15) $\frac{6}{2}$
(16) $\frac{3+5}{4+5}$
(17) $\frac{car}{cat}$
(18) $\frac{c+a+r}{c+a+t}$
(19) $\frac{8blah}{7blah}$

3.3. Dividing in $\mathbb{Q}$

" .. until one day, nothing happened,...... "

Gameplan 3.3
(1) *Multiplicative Inverses (again)*
(2) *What is 'divide'*
(3) *Practice*
(4) *Fractions of Fractions*

Multiplicative Inverses

We revisit multiplicative inverses with some new ideas. Recall we defined the multiplicative inverse of 5 to be $\frac{1}{5}$. We now consider finding the multiplicative inverse of a fraction like $\frac{3}{5}$. We contend that the multiplicative inverse of $\frac{3}{5}$ is $\frac{5}{3}$, (sometimes called it's reciprocal). Recall the definition of multiplicative inverses are pair that when multiplied together yield 1. We check to see that this happens here. If $\frac{3}{5} \cdot \frac{5}{3} = 1$, then we will know for sure that these are inverses.

$$\begin{aligned} \frac{3}{5} \cdot \frac{5}{3} &= \frac{3 \cdot 5}{5 \cdot 3} && \text{MAT} \\ &= \frac{15}{15} && \text{TT} \\ &= 1 && \text{JOT} \end{aligned}$$

Indeed, we have proven that $\frac{3}{5} \cdot \frac{5}{3} = 1$. Of course there is nothing special about 3 and 5, the same proof would have worked for any fraction. The multiplicative inverse for any fraction $\frac{a}{b}$, where $b \neq 0$ is always the reciprocal, $\frac{b}{a}$. We will make this a theorem, and because it tells us what the inverse of a fraction is, we will also call it the multiplicative inverse theorem [Minv]. In other words, it used to be that we only knew inverses of non-zero integers. We knew that if we just flip them with a one on top, we would have the inverse of any non-zero integer. We now find out that a similar principle works to find inverses of rational numbers. Namely, we need only flip a fraction to obtain the multiplicative inverse. Because of the similarity, we call both of these Multiplicative Inverses [MInv]. Also recall a fancy way of writing 'the multiplicative inverse of a' is just a^{-1}. So another way of writing $\left(\frac{3}{5}\right)^{-1}$, which is the multiplicative inverse of $\frac{3}{5}$ is to write $\frac{5}{3}$. A couple examples are in order.

(1) $5 \cdot \frac{1}{5} = 1$ [Minv (the old definition)]
(2) $\frac{3}{5} \cdot \frac{5}{3} = 1$ [Minv (the new theorem)]
(3) $\frac{13}{7} \cdot \frac{7}{13} = 1$ [Minv]
(4) $\left(\frac{7}{8}\right)^{-1} = \frac{8}{7}$ [Minv]
(5) $\frac{32}{71} \cdot \frac{71}{32} = 1$ [Minv]
(6) $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$ [Minv]

(7) $13 \cdot \frac{1}{13} = 1$ [Minv]

(8) $\left(\frac{3}{5}\right)^{-1} = \frac{5}{3}$ [Minv]

WHAT IS 'TO DIVIDE'

We will give two definitions, one in plain words and one using math symbols. Pretend this is the very first time you ever see 'divide.' *Divide* is a binary operation defined on any number pair of number (the second number non-zero) by the following. Using words;

$$a \div b = \ a \text{ times the multiplicative inverse of } b$$

Using math symbols;

$$a \div b = a \cdot b^{-1} \qquad \text{def of } \div$$

This definition should feel like a perfect fit. After all we defined 'subtraction' as 'adding the additive inverse.' So it should make perfect sense to define 'division' as 'multiplying by the multiplicative inverse'. The rest is *duck soup!* let the figuring begin.

PRACTICE

(1) Divide $\frac{3}{5} \div 7$

Solution:

$$\begin{aligned}
\frac{3}{5} \div 7 = & \quad \text{given} \\
&= \frac{3}{5} \cdot 7^{-1} && \text{def } \div \\
&= \frac{3}{5} \cdot \frac{1}{7} && \text{Minv (or neg Expo)} \\
&= \frac{3}{35} && \text{MAT, TT}
\end{aligned}$$

(2) Divide $\frac{3}{5} \div \frac{2}{7}$

Solution:

$$
\begin{aligned}
\frac{3}{5} \div \frac{2}{7} = & \quad \text{given} \\
&= \frac{3}{5} \cdot \left(\frac{2}{7}\right)^{-1} && \text{def } \div \\
&= \frac{3}{5} \cdot \frac{7}{2} && \text{Minv} \\
&= \frac{21}{10} && \text{MAT, TT}
\end{aligned}
$$

(3) Divide $\frac{3}{5} \div \frac{3}{8}$

Solution:

$$
\begin{aligned}
\frac{3}{5} \div \frac{3}{8} = & && \text{given} \\
&= \frac{3}{5} \cdot \left(\frac{3}{8}\right)^{-1} && \text{def } \div \\
&= \frac{3}{5} \cdot \frac{8}{3} && \text{Minv} \\
&= \frac{24}{15} && \text{MAT, TT}
\end{aligned}
$$

(simplify if you feel like doing so...)

Fractions of Fractions

Here is another look at the definition of division. To divide means to multiply by the inverse, then

$$
\begin{aligned}
a \div b &= a \cdot b^{-1} && \text{def of } \div \\
&= \frac{a}{b} && \text{def of } \frac{a}{b}
\end{aligned}
$$

This using both definitions, def $\div$ + def of $\frac{a}{b}$, we can turn every division into a fraction! Even better if we read backwards on the above argument, we can turn every fraction into a division problem! Observe...

$$\begin{aligned} \frac{3}{5} &= 3 \cdot \frac{1}{5} && \text{def of } \frac{a}{b} \\ &= 3 \div 5 && \text{def of } \div \end{aligned}$$

There is nothing special about 3 and 5. The argument will work for any numbers where the denominator is not zero. Because, this used two definition, the definition of fractions, and the definition of divide, we will call this the '*definition of divide and definition of fractions theorem*' or [Def $\div$, $\frac{a}{b}$]. Using it we can turn *every* fraction into a division problem. Observe:

(1) $\frac{5}{7} = 5 \div 7$ [Def $\div$, $\frac{a}{b}$]
(2) $\frac{2}{3} = 2 \div 3$ [Def $\div$, $\frac{a}{b}$]
(3) $\frac{cat}{rock} = cat \div rock$ [Def $\div$, $\frac{a}{b}$]

The value in turning every fraction into a division problem can be easily appreciated when dealing with complex fractions, or fractions of fractions. We continue with the same theme as above;

(1) $\frac{stuff}{blah} = stuff \div blah$ [Def $\div$, $\frac{a}{b}$]
(2) Simplify $\frac{\frac{3}{5}}{\frac{8}{13}}$

Solution:

$$\begin{aligned} \frac{\frac{3}{5}}{\frac{8}{13}} &= \frac{3}{5} \div \frac{8}{13} && \text{Def } \div, \frac{a}{b} \\ &= \frac{3}{5} \cdot \left(\frac{8}{13}\right)^{-1} && \text{Def } \div \\ &= \frac{3}{5} \cdot \frac{13}{8} && \text{Minv} \\ &= \frac{39}{40} && \text{MAT, TT} \end{aligned}$$

(3) Simplify $\frac{\frac{2}{3}}{\frac{5}{7}}$

Solution:

$$
\begin{aligned}
\frac{\frac{2}{3}}{\frac{5}{7}} &= \frac{2}{3} \div \frac{5}{7} && \text{Def } \div,\ \frac{a}{b} \\
&= \frac{2}{3} \cdot \left(\frac{5}{7}\right)^{-1} && \text{Def } \div \\
&= \frac{2}{3} \cdot \frac{7}{5} && \text{Minv} \\
&= \frac{14}{15} && \text{MAT, TT}
\end{aligned}
$$

Excercises 3.3

(1) $\frac{2}{3} \div \frac{8}{3}$
(2) $\frac{7}{13} \div \frac{5}{7}$
(3) $\frac{-7}{3} \div \frac{2}{5}$
(4) $\frac{-20}{-30} \div \frac{-8}{1}$
(5) $\frac{-20}{-30} \div 1$
(6) $\frac{\frac{-20}{-30}}{\frac{-20}{-30}}$
(7) $\frac{\frac{7}{3}}{\frac{5}{2}}$
(8) $\frac{\frac{7}{13}}{\frac{5}{7}}$
(9) $\frac{-7}{3} \div \frac{\frac{2}{3}}{\frac{2}{7}}$

3.4. Adding in $\mathbb{Q}$

" .. until one day, nothing happened,...... "

Gameplan 3.4
(1) *Idea*
(2) *Adding: w/ like denominator*
(3) *Adding: w/ different denominator*
(4) *Subtracting Fractions*

Idea

We are now ready to start adding some fractions. We first learn to add fractions with the same denominator and then learn to add fractions with different denominator.

In preparations, we will take a moment to review some of the important axioms/defintions we will be using here. First, we will use the definition of a fractions $\frac{a}{b}$, which it $a \cdot \frac{1}{b}$. In addition, we will be using the Multiplicative Identity axiom, which says for any number *blah*, we have $1 \cdot blah$. Moreover, we will see the Distributive Law again working it's magic. You will see these axioms working in flawless harmony to bring you precise and powerful reasoning through the addition of any pair of fractions.

Adding: Like Denominator

As usual we begin with some numerical examples that lead into the general proof. First, the examples,

(1) $\frac{7}{3} + \frac{5}{3}$

Solution:

$$\begin{aligned}
\frac{7}{3} + \frac{6}{3} &= 7 \cdot \frac{1}{3} + 6 \cdot \frac{1}{3} && \text{def } \frac{a}{b} \\
&= (7 + 6)\, \frac{1}{3} && \text{DL (amaizing!)} \\
&= 13 \cdot \frac{1}{3} && \text{AT} \\
&= \frac{13}{3} && \text{def } \frac{a}{b}
\end{aligned}$$

(2) $\frac{2}{5} + \frac{7}{5}$

Solution:

$$\begin{aligned}
\frac{2}{5}+\frac{7}{5} &= 2\cdot\frac{1}{5}+7\cdot\frac{1}{5} && \text{def } \frac{a}{b}\\
&= (2+7)\,\frac{1}{5} && \text{DL}\\
&= 9\cdot\frac{1}{5} && \text{AT}\\
&= \frac{9}{3} && \text{def } \frac{a}{b}
\end{aligned}$$

The pattern is clear. Adding fractions with like denominator only requires that we add the numerators. It should make sense on many different levels. Two *cats* plus seven *cats* is equal to nine *cats*. Then, two $fifths$ plus seven $fifths$ should be equal to nine $fifths$, as *proven* in the example above.

We generalize, name and prove this theorem. The name is *add the tops theorem [ATT], and the proof is given below.*

$$\frac{a}{b}+\frac{c}{b}=\frac{(a+c)}{b}$$

$$\begin{aligned}
&\frac{a}{b}+\frac{c}{b}= && \text{Suppose this is given with } b\neq 0\\
&\quad = a\cdot\frac{1}{b}+c\cdot\frac{1}{b} && \text{Def of } \frac{a}{b}\\
&\quad = (a+c)\frac{1}{b} && \text{DL}\\
&\quad = \frac{(a+c)}{b} && \text{Def of } \frac{a}{b}
\end{aligned}$$

Thus if $b\neq 0$ then $\dfrac{a}{b}+\dfrac{c}{b}=\dfrac{(a+c)}{b}$

Armed with this theorem, we can now add these type of fractions with the greatest of ease. Observe;

(1) $\frac{7}{3}+\frac{4}{3}$

Solution:

$$\begin{aligned}
\frac{7}{3}+\frac{4}{3} &= \frac{7+4}{3} && \text{ATT}\\
&= \frac{11}{3} && \text{AT}
\end{aligned}$$

(2) $\frac{8}{3}+\frac{-4}{3}$

Solution:

$$
\begin{aligned}
\frac{8}{3}+\frac{-4}{3} &= \frac{8+-4}{3} && \text{ATT}\\
&= \frac{4+4+-4}{3} && \text{AT}\\
&= \frac{4+0}{3} && \text{Ainv}\\
&= \frac{4}{3} && \text{AId}
\end{aligned}
$$

(3) $\frac{7}{5}+\frac{4}{5}$

Solution:

$$
\begin{aligned}
\frac{7}{5}+\frac{4}{5} &= \frac{7+4}{5} && \text{ATT}\\
&= \frac{11}{5} && \text{AT}
\end{aligned}
$$

ADDING: UNLIKE DENOMINATOR

Once we've learned to add fractions with like denominators, we are ready to take on the ones with different denominators. There is no new theorems to learn here. The axioms will lead us to the correct answer. Observe;

(1) Add $\frac{2}{3}+\frac{3}{5}$

Solution:

We will tackle the problem by first re-writing the fractions so that they share a common denominator. The common denominator will usually be the LCM of each of the denominators. In this case, $\text{LCM}(3,5)=15$. So we will try to rewrite each one with denominator 15.

$$\begin{aligned}
\frac{2}{3}+\frac{3}{5} &= \frac{2}{3}\cdot 1+\frac{3}{5}\cdot 1 && \text{Mid}\\
&= \frac{2}{3}\cdot\frac{5}{5}+\frac{3}{5}\cdot\frac{3}{3} && \text{JOT}\\
&= \frac{2\cdot 5}{3\cdot 5}+\frac{3\cdot 3}{5\cdot 3} && \text{MAT}\\
&= \frac{10}{15}+\frac{9}{15} && \text{TT}\\
&= \frac{10+9}{15} && \text{ATT}\\
&= \frac{19}{15} && \text{AT}
\end{aligned}$$

(2) Add $\frac{2}{7}+\frac{3}{5}$

Solution:

$$\begin{aligned}
\frac{2}{7}+\frac{3}{5} &= \frac{2}{7}\cdot 1+\frac{3}{5}\cdot 1 && \text{Mid}\\
&= \frac{2}{7}\cdot\frac{5}{5}+\frac{3}{5}\cdot\frac{7}{7} && \text{JOT}\\
&= \frac{2\cdot 5}{7\cdot 5}+\frac{3\cdot 7}{5\cdot 7} && \text{MAT}\\
&= \frac{10}{35}+\frac{21}{35} && \text{TT}\\
&= \frac{10+21}{35} && \text{ATT}\\
&= \frac{31}{35} && \text{AT}
\end{aligned}$$

(3) Add $\frac{2}{7}+3$

Solution:

$$
\begin{aligned}
\frac{2}{7} + 3 &= \frac{2}{7} + \frac{3}{1} && \text{OUT} \\
&= \frac{2}{7} + \frac{3}{1} \cdot 1 && \text{Mid} \\
&= \frac{2}{7} + \frac{3}{1} \cdot \frac{7}{7} && \text{JOT} \\
&= \frac{2}{7} + \frac{3 \cdot 7}{1 \cdot 7} && \text{MAT} \\
&= \frac{2}{7} + \frac{21}{7} && \text{TT} \\
&= \frac{2 + 21}{7} && \text{ATT} \\
&= \frac{23}{7} && \text{AT}
\end{aligned}
$$

(4) Add $\frac{3}{12} + \frac{5}{8}$

Solution:

$$
\begin{aligned}
\frac{3}{12} + \frac{5}{8} &= \frac{3}{12} \cdot 1 + \frac{5}{8} \cdot 1 && \text{Mid} \\
&= \frac{3}{12} \cdot \frac{2}{2} + \frac{5}{8} \cdot \frac{3}{3} && \text{JOT} \\
&= \frac{3 \cdot 2}{12 \cdot 2} + \frac{5 \cdot 3}{8 \cdot 3} && \text{MAT} \\
&= \frac{6}{24} + \frac{15}{24} && \text{TT} \\
&= \frac{6 + 15}{24} && \text{ATT} \\
&= \frac{21}{24} && \text{AT (simplify optional)} \\
&= \frac{3 \cdot 7}{3 \cdot 8} && \text{TT} \\
&= \frac{3}{3} \cdot \frac{7}{8} && \text{MAT} \\
&= 1 \cdot \frac{7}{8} && \text{JOT} \\
&= \frac{7}{8} && \text{Mid}
\end{aligned}
$$

Subtracting Fractions

We have already defined subtraction as 'adding the inverse'. We will continue to use this definition to subtract fractions.

Subtract $\frac{2}{3} - \frac{5}{4}$

Solution:

$$\begin{aligned}
\frac{2}{3} - \frac{5}{4} &= \frac{2}{3} + -\frac{5}{4} && \text{def } a-b\\
&= \frac{2}{3} + \frac{-5}{4} && \text{NWW}\\
&= \frac{2}{3}\cdot 1 + \frac{-5}{4}\cdot 1 && \text{Mid}\\
&= \frac{2}{3}\cdot\frac{4}{4} + \frac{-5}{4}\cdot\frac{3}{3} && \text{JOT}\\
&= \frac{2\cdot 4}{3\cdot 4} + \frac{-5\cdot 3}{4\cdot 3} && \text{MAT}\\
&= \frac{8}{12} + \frac{-15}{12} && \text{TT, NPT}\\
&= \frac{8 + -15}{12} && \text{ATT}\\
&= \frac{8 + -8 + -7}{12} && \text{N+NT}\\
&= \frac{0 + -7}{12} && \text{Ainv}\\
&= \frac{-7}{12} && \text{Aid}
\end{aligned}$$

EXCERCISES 3.4

(1) $\frac{-9}{1} + \frac{4}{-5}$
(2) $\frac{7}{1} + \frac{6}{9}$
(3) $\frac{7}{1} + \frac{-1}{-4}$
(4) $\frac{3}{5} - \frac{2}{-2}$
(5) $\frac{5}{4} + \frac{2}{-4}$
(6) $\frac{5}{-1} - \frac{-4}{-1} + \frac{3}{-8}$
(7) $\frac{-2}{6} + \frac{-6}{-9} + \frac{-5}{3}$
(8) $\frac{6}{-7} + \frac{-2}{5} + \frac{3}{6}$
(9) $\frac{-3}{2} + \frac{-7}{1} + \frac{0}{1}$
(10) $\frac{6}{2} + \frac{-1}{8} + \frac{7}{7}$
(11) $\frac{6}{2} + \frac{1}{2} + \frac{7}{6}$
(12) $\frac{3}{2x} + \frac{5}{x}$
(13) $\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{3}}}}$
(14) $\frac{2}{2+\frac{1}{3+\frac{1}{4+\frac{1}{3}}}}$

CHAPTER 4

Polynomials

4.1. Exponential Review

"until one day... nothing happened"

Gameplan 4.1
(1) *Review Expo Defs/Thms*
(2) *Practice*
(3) *CSi*

RECALL FAMOUS EXPONENT DEFINITIONS

(1) Positive Exponents: Recall we have already seen a definition for positive integer exponents. Namely, any number x raised to the 5th power means;

$$x^5 = x \cdot x \cdot x \cdot x \cdot x$$

(2) Negative Exponents: Recall we have already seen the definition for negative integer exponents. Namely, any number x raised to the -5 is equivalent to;

$$x^{-5} = \frac{1}{x^5}$$

(3) Zero Exponent: Recall the definition for a zero exponent has been defined for any non-zero base number x as;

$$x^0 = 1$$

RECALL FAMOUS EXPONENT THEOREMS

(1) *Just Add Exponents [JAE]*: Recall we can multiply any two numbers that have the same base by simply adding the exponents. Consider the following examples.

(a)

$$x^3 \cdot x^6 = x^9 \qquad [JAE]$$

(b)

$$x^{-3} \cdot x^6 = x^3 \qquad [JAE]$$

(2) *Power To Power [P2P]*: Recall we can simplify any number raised to a power, that quantity raised to another power using *[P2P]*.

(a)

$$\left(x^3\right)^6 = x^{18} \qquad [P2P]$$

(b)

$$\left(x^{-2}\right)^5 = x^{-10} \qquad [P2P]$$

A NEW EXPONENT THEOREM: [CSIT]

In this theorem we address fractions that have terms with negative exponents. Consider the fraction

$$\frac{Ax^{-5}}{B}$$

We can simplify it by using the definition of negative exponents. When it's all said and done we find the x^{-5} goes to the denominator and in doing so we *change the sign* of the exponent. Here is all the steps that go in this process.

$$\begin{aligned}
\frac{Ax^{-5}}{B} &= \frac{Ax^{-5}}{B \cdot 1} && \text{Mid} \\
&= \frac{A}{B} \cdot \frac{x^{-5}}{1} && \text{MAT} \\
&= \frac{A}{B} \cdot x^{-5} && \text{OUT} \\
&= \frac{A}{B} \cdot \frac{1}{x^5} && \text{neg Expo} \\
&= \frac{A}{Bx^5} && \text{MAT}
\end{aligned}$$

A similar argument shows that a term with a negative exponent in the denominator can be moved to the numerator by changing the sign of the exponent. In other words, if we start with a fraction of the type

$$\frac{A}{Bx^{-5}}$$

we can move the x^{-5} to the numerator while changing the sign of the exponent to x^5. It is important to note this only works if the denominator is B times x^{-5}. In particular, it would not work if the denominator was $B + x^{-5}$. It can only work for multiplication, Bx^{-5}. We detail the argument below.

$$
\begin{aligned}
\frac{A}{Bx^{-5}} &= \frac{A}{Bx^{-5}} \cdot 1 && \text{MId} \\
&= \frac{A}{Bx^{-5}} \cdot \frac{x^5}{x^5} && \text{JOT} \\
&= \frac{Ax^5}{Bx^5x^{-5}} && \text{MAT} \\
&= \frac{Ax^5}{Bx^{5+-5}} && \text{JAE} \\
&= \frac{Ax^5}{Bx^0} && \text{Minv} \\
&= \frac{Ax^5}{B \cdot 1} && \text{0-Expo} \\
&= \frac{Ax^5}{B} && \text{MId}
\end{aligned}
$$

The result is the same as before; crossing the numerator/denominator line causes the sign of the exponent to change. We shall call this the *Change the Sign Theorem [CSiT].*

PRACTICE

(1) Simplify $x^3 \cdot x^5$?

$$
\begin{aligned}
x^3 \cdot x^5 &= x^{3+5} && \text{JAE} \\
&= x^8 && \text{AT}
\end{aligned}
$$

(2) Simplify $y \cdot y^3$

$$
\begin{aligned}
y \cdot y^3 &= y^1 \cdot y^3 && \text{+Expo} \\
&= y^{1+3} && \text{JAE} \\
&= y^4 && \text{AT}
\end{aligned}
$$

(3) Simplify $x^3y^2x^2y$

$$
\begin{aligned}
x^3y^2x^2y &= x^3x^2y^2y && \text{CoLM} \\
&= x^3x^2y^2y^1 && \text{+Expo} \\
&= x^5y^2 && \text{JAE}
\end{aligned}
$$

(4) Simplify x^3x^{-2}

$$\begin{aligned} x^3x^{-2} &= x^{3+-2} && \text{JAE} \\ &= x^1 && \text{BI} \\ &= x && \text{+Expo} \end{aligned}$$

(5) Simplify x^3x^{-3}

$$\begin{aligned} x^3x^{-3} &= x^{3+-3} && \text{JAE} \\ &= x^0 && \text{AInv} \\ &= 1 && \text{0-Expo} \end{aligned}$$

(6) Simplify x^3x^{-2}

$$\begin{aligned} x^3x^{-7} &= x^{(3+-7)} && \text{JAE} \\ &= x^{-4} && \text{BI} \\ &= \frac{1}{x^4} && \text{+Expo} \end{aligned}$$

(7) Simplify $x^3y^{-4}x^{-2}y^2$

$$\begin{aligned} x^3y^{-4}x^{-2}y^2 &= x^3x^{-2}y^{-4}y^2 && \text{CoLM} \\ &= x^{(3+-2)}y^{(-4+2)} && \text{JAE} \\ &= x^1 \cdot y^{-2} && \text{BI} \\ &= x \cdot \frac{1}{y^2} && \text{+Expo, neg Expo} \\ &= \frac{x}{y^2} && \text{def } a/b \end{aligned}$$

(8) Simplify $\frac{yx^{-5}}{x^4}$

$$\begin{aligned} \frac{yx^{-5}}{x^4} &= \frac{y}{x^4x^5} && \text{CSiT} \\ &= \frac{y}{x^9} && \text{JAE} \end{aligned}$$

(9) Simplify $\frac{x^8}{yx^{-5}}$

$$\begin{aligned} \frac{x^8}{yx^{-5}} &= \frac{x^8x^5}{y} && \text{CSiT} \\ &= \frac{x^{13}}{y} && \text{JAE} \end{aligned}$$

(10) Simplify $\frac{x^8y^{-6}}{yx^{-5}}$

$$\begin{aligned}\frac{x^8y^{-6}}{yx^{-5}} &= \frac{x^8x^5}{y^6y} && \text{CSiT}\\ &= \frac{x^{13}}{y^7} && \text{JAE}\end{aligned}$$

c

Excercises 4.1

(1) $x^{-2}y^{-1}x^{-4}y^{-4}$
(2) $x^{-1}yx^{-1}y$
(3) $x^{-4}y^{-2}x^2y^2$
(4) $x^{-1}yxy^{-1}$
(5) $x^2y^{-1}xy^3$
(6) $\frac{x^2}{x^{-7}y^{-10}}$
(7) $\frac{2}{x^{-7}y^{-10}}$
(8) $\frac{x^{-3}y^{-7}}{x^{-7}y^{-10}}$
(9) $\frac{x^{-3}z^{-7}}{x^{-2}y^{-10}}$
(10) $x^3y^2x^{-2}y^2$
(11) $x^{-1}yx^{-2}y^2$
(12) $xyx^{-4}y^{-4}$
(13) $x^{-4}y^{-4}x^2y^{-1}$
(14) $x^{-4}y^3x^{-1}y^{-1}$
(15) $x^{-2}y^3x^3y^{-4}z^0$

4.2. Polynomial Introduction

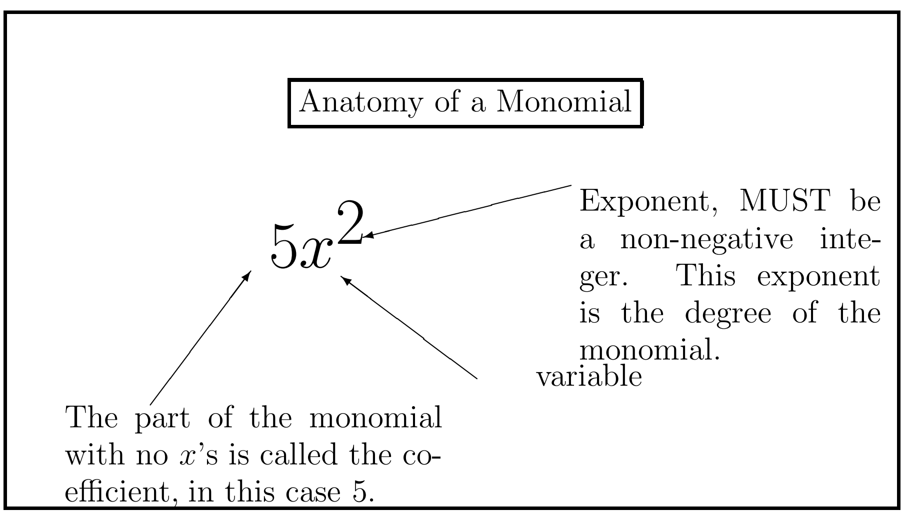
"a rose is a rose is a rose... Reflexive property"

Gameplan 4.2

(1) *Monomial*
(2) *Binomial*
(3) *Polynomial*
(4) *Polynomial...NoT*

The Anatomy of a Monomial

A monomial is generally made up of two parts. These parts are always multiplied, thus we can refer to them as factors. One of the factors is *the variable*, usually x, raised to an exponent. The other factor is usually called *the coefficient.* The exponent is also referred to as *the degree* of the monomial. Below we illustrate a typical monomial.



Although, most of the monomials we will encounter will be very much like the monomial above, there can be many variations. Consider the following monomial.

$$3b\pi^2x^7$$

In this case, there are two variables, x and b. If we consider this as a *monomial in x*, then the degree in x is 7, while the coefficient is the rest of the stuff, $3b\pi^2$. On the other hand, if we look at it as a monomial in b, we can rewrite it as

$$3\pi^2x^7b.$$

As a monomial in b, the coefficient is $3\pi^2x^7$ while the degree in b is 1.

It should be emphasized that monomials contain *only* non-negative integer exponents on the variable. Thus, none of the following terms will not be consider a *monomial.*

$$3x^{-2}, \qquad 3x^{\frac{2}{3}}$$

Moreover, monomials may contain *just* the variable, or *the product* of the variable times something else called the coefficient. The following is not a monomial

$$3 + x^2.$$

In fact, addition/subtraction signs *separate* monomials. Which leads us to the definition of a *binomial.*

Binomials

Simply put, a binomial is the sum or difference of two monomials. The *degree of the binomial* is determined by *the higher* of the degrees for each of the monomials. As before, each piece of the binomial has a special name. Consider the following example.

$$3x^5 + 13x^2$$

First, note this is *the sum of two monomials*, thus it is indeed a bonified *binomial.* The first term is a degree 5 monomial, while the other term is a degree 2 monomial. Since the higher of the two degrees is 5, we say *the degree of the binomial* is 5. In addition, each coefficient has its own name. The 3 is called *the degree-five coefficient,* while the degree-two coefficient is 13.

Polynomials

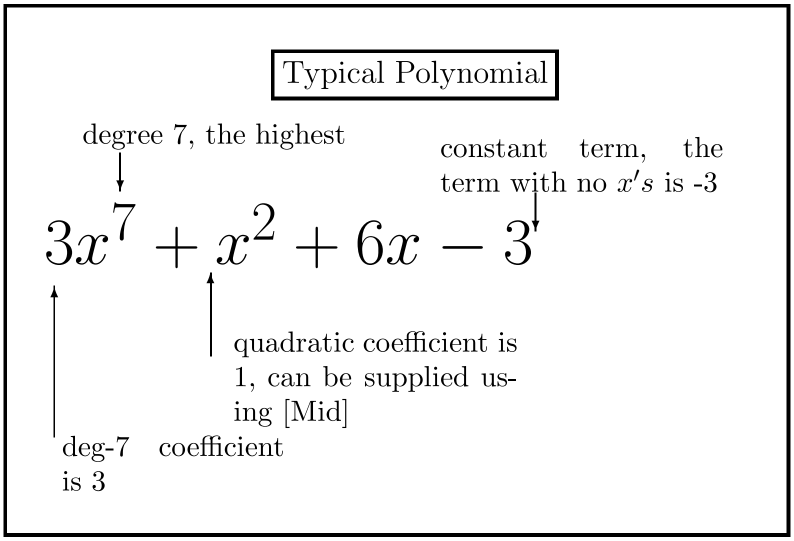
As the name suggests, a *polynomial* is the sum/difference of many monomials. The same conventions discussed for the binomials are used for polynomials. The degree of the polynomial is defined to be the highest degree of each of the monomials. Consider the following polynomial.

$$3x^7 + x^2 + 6x - 3$$

In this case, the degree of the polynomial is 7. Often, monomials appear to have no coefficient, such as 'x^2' above. We can readily supply a coefficient by using the multiplicative identity, which says

$$x^2 = 1x^2.$$

In other words, the degree two coefficient in the above polynomial is 1. Often, degree-two monomials are called *quadratic*, while degree-one polynomials are called *linear.* The term with no x's on it is often called the *constant term.* We summarize the different parts of a typical polynomial below.



Again we emphasize the few but crucial requirements to be a polynomial. A polynomial is the sum or difference of one or more monomials. Each monomial is made up of the product of a variable with a coefficient. The exponent on the variable must be a non-negative whole integer. The monomial of highest degree determines the degree of the polynomial. The entire family of polynomials that have variable x and rational number coefficients is represented by the symbol $\mathbb{Q}[x]$. The symbol $\mathbb{Z}[y]$ on the other hand would describe the family of all polynomials with *integer* coefficients and variable y.

Classroom Exercises

(1) Which is a polynomial and which is not?

(a) $3x^4 + 6x - \frac{1}{x}$
(b) $3x^2 + \sqrt{x} + 3$
(c) $5x^3 + 7x - \pi$
(d) $\frac{3x+5}{6x-1}$
(e) $\log(3x^3 + 8)$
(f) 5^x
(g) x^5
(h) $4x^4 - 23x^5 + x^{-3}$

(2) Determine the degree of the polynomial and name each coefficient.

$$3x^4 - 4x^2 + x - 5$$

(3) Determine the degree of the polynomial and name each coefficient.

$$-x^5 + 4x^2 + x - 5 + \pi$$

(4) Determine the degree of the polynomial and name each coefficient.

$$-3x^7 + (4 + \pi)x^3 + 5x + \sqrt{7} - 5 + \pi$$

4.3. Adding in $\mathbb{Q}[x]$

"a rose is a rose is a rose... Reflexive property"

Gameplan 4.3
(1) *Adding*
(2) *Subtracting*
(3) *Practice*

Adding

Having learned in great detail the art of multiplying, adding, subtracting, and dividing integers, we will often call on this experience to guide us as we learn how to add, multiply, divide, and factor *polynomials.* Of course, one big obvious difference between the world of integers, $\mathbb{Z}$, and and the world of polynomials, $\mathbb{Q}[x]$, is that there are variables in the the world of polynomials. However, traditionally, these variable have always represented numbers. This being said, we can assume if the variables represent numbers then they follow the number axioms. This for of reasonings leads us to adopt *almost all* the axioms we have had for integers and rational numbers for polynomials. Commutativity, Associativity, Distributive Law, and Exponent definitions are just some of the examples of axioms that hold true in the world of $\mathbb{Q}[x]$. There may be a couple of exceptions. For example, we do not have a *times table [TT]* to help us multiply $x \cdot x$ or $3 \cdot x$. The best we can do in this case is $x \cdot x = x^2\,[+Expo]$ and $3 \cdot x = 3x\,[RP]$. Additive inverses still work the same way, $x + -x = 0\,[Ainv]$. (However, the multiplicative inverse of a polynomial may not be a polynomial...think about it...)

The immediate goal is a modest one, to add polynomials. In a sentence, adding polynomials means collect like terms. Let the adding begin.

Examples

(1) $(2x + 3) + (5x + 3)$

Solution:

$$\begin{aligned}
(2x+3)+(5x+3) &= 2x+3+5x+3 && \text{AlA}\\
&= 3+3+2x+5x && \text{CoLA}\\
&= 3+3+(2+5)x && \text{DL}\\
&= 6+7x && \text{AT}
\end{aligned}$$

(2) $(1 + 2x) + (2 + 3x)$

Solution:

$$\begin{aligned}(1+2x)+(2+3x) &= 1+2x+2+3x && \text{ALA}\\ &= 1+2+2x+3x && \text{CoLA}\\ &= 1+2+(2+3)x && \text{DL}\\ &= 3+5x && \text{AT}\end{aligned}$$

(3) $(1+2x+x^2)+(2+^-3x+^-3x^2)$

Solution:

$$\begin{aligned}(1+2x+x^2)+(2+^-3x+^-3x^2) &= 1+2x+x^2+2+^-3x+^-3x^2 && \text{ALA}\\ &= 1+2+2x+^-3x+^-3x^2+x^2 && \text{CoLA}\\ &= 1+2+2x+^-3x+^-3x^2+1\cdot x^2 && \text{Mid}\\ &= 1+2+(2+^-3)x+(^-3+1)x^2 && \text{DL}\\ &= 3+^-1x+^-2x^2 && \text{BI}\\ &= 3-x-2x^2 && \text{BI}\end{aligned}$$

(4) $(1+2x+x^2)-(2+^-3x+^-3x^2)$

Solution:

$$\begin{aligned}&(1+2x+x^2)-(2+^-3x+^-3x^2) && \text{given}\\ &= (1+2x+x^2)+^-(2+^-3x+^-3x^2) && \text{def } a-b\\ &= (1+2x+x^2)+^-1(2+^-3x+^-3x^2) && \text{MT}\\ &= (1+2x+x^2)+^-1\cdot 2+^-1\cdot ^-3x+^-1\cdot ^-3x^2 && \text{DL}\\ &= (1+2x+x^2)+^-2+3x+3x^2 && \text{NPT, NNT}\\ &= 1+2x+x^2+^-2+3x+3x^2 && \text{ALA}\\ &=^-2+1+2x+3x+3x^2+x^2 && \text{CoLA}\\ &=^-2+1+2x+3x+3x^2+1\cdot x^2 && \text{Mid}\\ &=^-2+1+(2+3)x+(3+1)x^2 && \text{DL}\\ &=^-1+5x+4x^2 && \text{BI}\end{aligned}$$

Classroom Exercises

(1) $(3x^2 + 2x) + (x + 5x^2 + \pi)$

(2) $(3 - 6x) + (2x + 5x^3)$

(3) $(5x^2 - 4x) + (\pi x + 5x^2)$

(4) $(x + 3) - (2x + 5)$

(5) $x^3 + 2x$

(6) $x^3 + 5x^3$

Excercises 4.3

(1) $(3-4x)+(3x+5x^3)$
(2) $(3x^2-4x)+(\pi x+5x^2)$
(3) $(3x^2-4x+2)-(3x+5x^2+^-2)$
(4) $(x+3)-(3x+5)$
(5) x^3+3x
(6) x^3+3x^3
(7) x^2+x
(8) x^2+x^3
(9) x^2-x
(10) x^2-x^2
(11) $2x-x$
(12) $5x-x$

4.4. Multiplying in $\mathbb{Q}[x]$

"a rose is a rose is a rose... Reflexive property"

Gameplan 4.4
(1) *Multiplying by Monomial*
(2) *Multiplying by Binomial*
(3) *FOIL*
(4) *Kindergarten Method*
(5) *The Famous*

Multiplying by Monomial

With all the ideas developed and nourished, nothing stands in the way. We are ready to multiply any monomial by any polynomial. The only thing we need now is courage and practice. Ready... set...go!

(1) Example (monomial x monomial): Multiply $(23x^6)(3x^5)$

Solution:

$$\begin{aligned} (23x^6)(3x^5) &= 23x^6 \cdot 3x^5 && \text{ALM} \\ &= 23 \cdot 3x^6x^5 && \text{CoLM} \\ &= 69x^6x^5 && \text{TT} \\ &= 69x^{11} && \text{JAE} \end{aligned}$$

(2) Example (monomial x monomial): Multiply $(3x^6)(2x)$

Solution:

$$\begin{aligned} (3x^6)(2x) &= 3x^6 \cdot 2x && \text{ALM} \\ &= 3 \cdot 2x^6x && \text{CoLM} \\ &= 6x^6x && \text{TT} \\ &= 6x^6x^1 && \text{+Expo} \\ &= 6x^7 && \text{JAE} \end{aligned}$$

(3) Example (monomial x polynomial): Multiply $3x^2(2x^5 + 3x^2 +^- 7)$

Solution:

$$\begin{aligned} 3x^2(2x^5+3x^2+^-7) &= 3x^2\cdot 2x^5+3x^2\cdot 3x^2+3x^2\cdot ^-7 && \text{DL}\\ &= 3\cdot 2x^2x^5+3\cdot 3x^2x^2+3\cdot ^-7x^2 && \text{CoLM}\\ &= 6x^2x^5+9x^2x^2+^-21x^2 && \text{TT, NPT}\\ &= 6x^7+9x^4+^-21x^2 && \text{JAE} \end{aligned}$$

(4) Example (polynomial x monomial): Multiply $(x^2+3x+^-7)3x^5$

Solution:

$$\begin{aligned} (x^2+3x+^-7)3x^5 &= x^2\cdot 3x^5+3x\cdot 3x^5+^-7\cdot 3x^5 && \text{DL}\\ &= 3x^2x^5+3\cdot 3xx^5+^-7\cdot 3x^5 && \text{CoLM}\\ &= 3x^2x^5+9xx^5+^-21x^5 && \text{TT,NPT}\\ &= 3x^2x^5+9x^1x^5+^-21x^5 && \text{+Expo}\\ &= 3x^7+9x^6+^-21x^5 && \text{JAT} \end{aligned}$$

(5) Example (monomial x monomial): Multiply $(23x^6)(3x^5)$

Solution:

$$\begin{aligned} (23x^6)(3x^5) &= 23x^6\cdot 3x^5 && \text{ALM}\\ &= 23\cdot 3x^6x^5 && \text{CoLM}\\ &= 69x^6x^5 && \text{TT}\\ &= 69x^{11} && \text{JAE} \end{aligned}$$

Multiplying by Binomial

With all the ideas developed and nourished, nothing stands in the way. We are ready to multiply any monomial by any polynomial. The only thing we need now is courage and practice. Ready... set...go!

(1) Example (binomial x polynomial): Multiply $(x^2+^-7)(3+x^3)$

Solution:

$$\begin{aligned}
(x^2 +^- 7)(3 + x^3) &= (x^2 +^- 7)3 + (x^2 +^- 7)x^3 && \text{DL} \\
&= x^2 \cdot 3 +^- 7 \cdot 3 + x^2x^3 +^- 7x^3 && \text{DL} \\
&= 3x^2 +^- 7 \cdot 3 + x^2x^3 +^- 7x^3 && \text{CoLM} \\
&= 3x^2 +^- 21 + x^2x^3 +^- 7x^3 && \text{NPT} \\
&= 3x^2 +^- 21 + x^5 +^- 7x^3 && \text{JAE}
\end{aligned}$$

(2) Example (binomial x polynomial): Multiply $(2x +^- 2)(x^2 + 3x - 1)$

Solution:

$$\begin{aligned}
(2x +^- 2)(x^2 + 3x - 1) &= (2x +^- 2)(x^2 + 3x +^- 1) && \text{Def } a - b \\
&= (2x +^- 2)x^2 + (2x +^- 2)3x + (2x +^- 2)^-1 && \text{DL} \\
&= 2xx^2 +^- 2x^2 + 2x \cdot 3x +^- 2 \cdot 3x + 2x \cdot^- 1 +^- 2 \cdot^- 1 && \text{DL} \\
&= 2x^1x^2 +^- 2x^2 + 2x^1 \cdot 3x^1 +^- 2 \cdot 3x + 2x \cdot^- 1 +^- 2 \cdot^- 1 && \text{+Expo} \\
&= 2x^1x^2 +^- 2x^2 + 2 \cdot 3x^1x^1 +^- 2 \cdot 3x + 2 \cdot^- 1x +^- 2 \cdot^- 1 && \text{CoLM} \\
&= 2x^3 +^- 2x^2 + 2 \cdot 3x^2 +^- 2 \cdot 3x + 2 \cdot^- 1x +^- 2 \cdot^- 1 && \text{JAE} \\
&= 2x^3 +^- 2x^2 + 6x^2 +^- 6x +^- 2x + 2 && \text{BI}
\end{aligned}$$

THE FOIL STORY

Multiplying binomials by binomials is a task to which you will be called upon very frequently. We take a moment here to consider the consequences and the patterns that arise from multiplying a binomial by a binomial. In fact, the pattern we will discover is commonly referred to as the FOIL method. Consider the following product.

$$(a + b)(c + d)$$

We already know how to multiply these binomials.

$$\begin{aligned}
(a + b)(c + d) &= a(c + d) + b(c + d) && \text{DL} \\
&= ac + ad + bc + bd && \text{DL}
\end{aligned}$$

The product result in four terms. The first term is obtained from the product of the **F**irst term from each binomial, the a and the c. The second term is obtained by multiplying the **O**utside terms, a and d. The next term is obtained from the **I**nside terms b and c, while the last term is obtained by multiplying the **L**ast terms of each of the binomials, b and d, thus the **FOIL** name on the idea. As proven above, FOIL is nothing more than the

Distributive Law applied a couple of times, thus an alternative name for FOIL could have been DL twice, or DL^2. In any case, we should take a moment to see how we use this to multiply binomials.

(1) Example (Foil on binomial x binomial): Multiply $(2x+3)(3x+1)$

Solution:

$$\begin{aligned}
(2x+3)(3x+1) &= 2x3x + 2x\cdot 1 + 3\cdot 3x + 3\cdot 1 && \text{FOIL}\\
&= 2\cdot 3xx + 2\cdot 1x + 3\cdot 3x + 3\cdot 1 && \text{CoLM}\\
&= 6xx + 2x + 9x + 3 && \text{TT}\\
&= 6x^2 + (2+9)x + 3 && \text{+Expo, DL}\\
&= 6x^2 + 11x + 3 && \text{AT}
\end{aligned}$$

(2) Example (Foil on binomial x binomial): Multiply $(x-3)(3x+1)$

Solution:

$$\begin{aligned}
(x-3)(3x+1) &= (x +^- 3)(3x+1) && \text{Def } a-b\\
&= x\cdot 3x + x\cdot 1 +^- 3\cdot 3x +^- 3\cdot 1 && \text{FOIL}\\
&= 3xx + 1\cdot x +^- 9x +^- 3 && \text{CoLM, NPT}\\
&= 3x^2 + (1 +^- 9)x +^- 3 && \text{+Expo, DL}\\
&= 3x^2 +^- 8x +^- 3 && \text{BI}
\end{aligned}$$

(3) Example (Famous! binomial x binomial): Multiply $(x-3)(x+3)$

Solution:

$$\begin{aligned}
(x-3)(x+3) &= (x +^- 3)(x+3) && \text{Def } a-b\\
&= xx + 3x +^- 3x +^- 3\cdot 3 && \text{FOIL}\\
&= x^2 + (3 +^- 3)x +^- 3\cdot 3 && \text{+Expo, DL}\\
&= x^2 + 0x +^- 9 && \text{Ainv, NPT}\\
&= x^2 +^- 9 && \text{OMT}\\
&= x^2 - 9 && \text{def a-b}
\end{aligned}$$

Keep in mind that the FOIL theorem works best to multiply a *binomial by a binomial.* FOIL may prove to be completely useless when the time comes to multiply, for example, *a binomial by a trinomial.* To multiply a binomial by a trinomial, we have to look back to the source. The FOIL came from the distributive law, and the distributive law will go much further than FOIL. In fact, the distributive law can be used to multiply *any* two polynomials in the universe. Without a doubt, sheer dominance is what comes to mind when we talk about the *Distributive Law* in the world of $\mathbb{Q}[x]$ under multiplication.

Classroom Exercises

(1) $3x^2(5x - 4x + 1)$

(2) $(2 + 3)(1 + 5)$

(3) $(3x^2 + 1)(5x - 4x + 1)$

(4) $(3x - 2)(5x - 4x + 1)$

(5) (famous) $(x - 1)(x^2 + x + 1)$

(6) (famous) $(x + y)(x + y)$

Excercises 4.4

(1) $3x^4(3x^2 + 5x^3 - 2x + 2)$
(2) $(7x^2 + 5x^3 - x + 5)2x^3$
(3) $x^2(7x^2 + 5x^3 - x + 5)$
(4) $(7x^2 + 5x^3 - x + 5)(2x^3 + 2)$
(5) $(7x^3 + 5x^2 - x + 5)\pi x$
(6) $(4 + 3x)(-2 + 3x^2)$
(7) $(-1 + 2x)(-2 + 4x)$
(8) $(1 + x)(1 + x)$ (Very Famous!)
(9) $(1 - x)(1 + x)$ (Very Famous!)
(10) $(4 + -2x)(2 + 2x^2)$
(11) $(a + b)^2$ (Very Famous!)
(12) $(a - b)^2$ (Very Famous!)
(13) $(a + b)^3$ (Very Famous!)
(14) $(a - b)^3$ (Very Famous!)
(15) $(2 + 3x)(2 - 3x)$
(16) $(2 + 3x)(2 + 3x)$
(17) $(2 - 3x)(2 - 3x)$
(18) $(3 + 4y)({}^-2 +^- 2y^2)$
(19) $(2 + 3x)(1 + x + x^2)$
(20) $(x - 1)(x^2 + x^2 + x + 1)$
(21) $(x - 1)(x^3 + x^2 + x^2 + x + 1)$
(22) $(x - 1)(x^4 + x^3 + x^2 + x^2 + x + 1)$

4.5. Dividing in $\mathbb{Q}[x]$

"a rose is a rose is a rose... Reflexive property"

Gameplan 4.5
(1) *Dividing by Monomial*
(2) *Long Division*

Dividing by Monomial

The method we will use to divide polynomials is called *long division.* To fully appreciate this method, we must dig up the memories and skills we used when dividing regular integers. Indeed, according to historians and experts, there was once upon a time when students divided using paper and pencil! Calculators where not widely used. Consider dividing with paper and pencil,

$$12345 \div 7.$$

Solution:

```
        1763
 7  )12345
      7
      53
      49
       44
       42
       23
        21
         3
```

We can conclude, $12345 \div 7$ is *1763 with reminder 3*, or $1763 + \frac{3}{7}$. The idea is to generalize this method for polynomials. As usual, as assume the crawling is easier than walking, thus we begin with some easy divisions (*monomial* $\div$ *monomial*), and progress into the no-division-can-stop-me state of mind.

Examples:

(1) (Monomial $\div$ Monomial) $6x^4 \div 2x$

Solution:

$$
\begin{aligned}
6x^4 \div 2x &= \frac{6x^4}{2x} && \text{Def } \div\text{, a/b} \\
&= \frac{3 \cdot 2x^3x^1}{2x} && \text{JAE,TT} \\
&= \frac{3x^3 \cdot 2x}{1 \cdot 2x} && \text{CoLM, +Expo, Mid} \\
&= \frac{3x^3}{1}\frac{2x}{2x} && \text{MAT} \\
&= \frac{3x^3}{1} && \text{JOT,Mid} \\
&= 3x^3 && \text{OUT}
\end{aligned}
$$

(2) (Monomial $\div$ Monomial) $16x^4 \div 2x^3$

Solution:

$$
\begin{aligned}
16x^4 \div 2x^3 &= \frac{16x^4}{2x^3} && \text{Def } \div\text{, a/b} \\
&= \frac{8 \cdot 2x^1x^3}{1 \cdot 2x^3} && \text{JAE,TT} \\
&= \frac{8x \cdot 2x^3}{1 \cdot 2x^3} && \text{CoLM, +Expo} \\
&= \frac{8x}{1}\frac{2x^3}{2x^3} && \text{MAT} \\
&= \frac{8x}{1} \cdot 1 && \text{JOT} \\
&= 8x && \text{OUT,Mid}
\end{aligned}
$$

(3) (Poly $\div$ Monomial) $(6x^3 + 4x^2 + 1) \div 2x$

Solution:

$$
\begin{aligned}
(6x^3+4x^2+1)\div 2x &= \frac{6x^3+4x^2+1}{2x} && \text{Def a/b, } \div \\
&= \frac{6x^3}{2x}+\frac{4x^2}{2x}+\frac{1}{2x} && \text{ATT} \\
&= \frac{2x\cdot 3x^2}{2x\cdot 1}+\frac{2x\cdot 2x}{1\cdot 2x}+\frac{1}{2x} && \text{JAE,CoLM,BI} \\
&= \frac{2x}{2x}\frac{3x^2}{1}+\frac{2x}{1}\frac{2x}{2x}+\frac{1}{2x} && \text{MAT} \\
&= 1\cdot\frac{3x^2}{1}+\frac{2x}{1}\cdot 1+\frac{1}{2x} && \text{JOT} \\
&= 3x^2+2x+\frac{1}{2x} && \text{Mid, OUT}
\end{aligned}
$$

We can conclude $(6x^3+4x^2+1)\div 2x = 3x^2+2x+\frac{1}{2x}$. Alternatively, we say $(6x^3+4x^2+1)\div 2x = 3x^2+2x$ with reminder 1. This method used the idea that *every division problem can be expressed as a fraction,* while making use of our fraction skills. While it has its virtues, it also has drawbacks. This method may not be very useful when we try to divide *polynomial by polynomial.* Ultimately, we will have to resort to *long division,* which is king in the world of $mathbbQ[x]$ under division. Here is a second look at the same problem above, solved using long division.

(4) (Poly ÷ Monomial, using long division) $(6x^3+4x^2+1)\div 2x$

Solution:

First we set it up as, we do when we divide integers using long division,

$$
\begin{array}{r} [s \quad \overline{\;\;1}tage = 1]6x^3+4x^2+12x \\ 1\big)\;\;1 \\ -\,1 \\ \hline 0 \end{array}
$$

then the idea is to see how many times $2x$ goes into the leading term $6x^3$. In other words we calculate $\frac{6x^3}{2x}$. We can do this by inspection [BI] or as example 1, above. In either case, we conclude $\frac{6x^3}{2x} = 3x^2$. This becomes the first part of the quotient.

$$
\begin{array}{r} [s \quad \overline{\;\;1}tage = 2]6x^3+4x^2+12x \\ 1\big)\;\;1 \\ -\,1 \\ \hline 0 \end{array}
$$

It is customary to try to keep columns ordered by degree. Observe the $3x^2$ was placed on the degree 2 column. The next step, as with integers, is to multiply the $3x^2$ by the divisor $2x$ and subtract it from the leading term $6x^3$. The result is shown,

$$[s \quad 1tage = 3]6x^3 + 4x^2 + 12x$$
$$1\overline{)\,1}$$
$$-1$$
$$0$$

We subtract and bring down the next term $4x^2$ to obtain:

$$[s \quad 1tage = 4]6x^3 + 4x^2 + 12x$$
$$1\overline{)\,1}$$
$$-1$$
$$0$$

We now ask how many times will $2x$ go into $4x^2$? That is, we find $\frac{4x^2}{2x} = 2x$. This becomes the second part of the quotient. We obtain

$$[s \quad 1tage = 5]6x^3 + 4x^2 + 12x$$
$$1\overline{)\,1}$$
$$-1$$
$$0$$

The next step is to multiply the $2x$ in the quotient by the divisor, $2x$ to obtain $4x^2$ which we write down below and subtract from $4x^2$ to obtain

$$[s \quad 1tage = 8]6x^3 + 4x^2 + 12x$$
$$1\overline{)\,1}$$
$$-1$$
$$0$$

Finally, we bring down the 1. Since $2x$ does not divide 1 evenly ($\frac{1}{2x}$ does not reduce) it is left as a remainder. We conclude with the expected, $(6x^3 + 4x^2 + 1) \div 2x = 3x^2 + 2x + \frac{1}{2x}$ OR $(6x^3 + 4x^2 + 1) \div 2x = 3x^2 + 2x$ with reminder 1. A couple more examples are in order

(5) $(3x^3 + 15x^2 - 6x - 12) \div (3x)$

Solution:

$$\begin{array}{rl} & x^2 + 5x \quad - 2 \\ 3x\big) & \overline{3x^3 + 15x^2 - 6x - 12} \\ & \underline{-3x^3} \\ & 15x^2 \\ & \underline{-15x^2} \\ & -6x \\ & \underline{6x} \end{array}$$

Finally we bring down the -12 and find the remainder is -12, thus $(3x^3 + 15x^2 - 6x - 12) \div (3x) = x^2 + 5x - 2 + \frac{-12}{3x}$

(6) $(3x^3 + 5x^2 - 16x - 2) \div (3x)$

Solution:

$$\begin{array}{r|l}
 & x^2 \;\; + \tfrac{5}{3}x - \tfrac{16}{3} \\
\hline
3x & 3x^3 + 5x^2 - 16x \;\; - 2 \\
 & \underline{-\,3x^3} \\
 & \qquad\quad 5x^2 \\
 & \qquad\quad \underline{-\,5x^2} \\
 & \qquad\qquad\quad -\,16x \\
 & \qquad\qquad\quad \underline{\;\;\;16x}
\end{array}$$

Finally we bring down the (remainder) -2 and conclude, by *Long Division [LD]* $(3x^3 + 5x^2 - 16x - 2) \div (3x) = x^2 + \frac{5}{3}x + \frac{-16}{3} + \frac{-2}{3x}$

Classroom Exercises

(1) (do it by writing as a fraction)$51x^7 \div 3x^2$

(2) (do it by writing as a fraction) $121x^3 \div 11x$

(3) (do it by writing as a fraction)$15x^5 \div 3x^2$

(4) (do it by writing as a fraction)$(15x^5 + 121x^3 + 51x^7) \div 3x^2$

(5) (do it by long division)$(15x^5 + 121x^3 + 51x^7) \div 3x^2$

(6) (do it by long division)$(15x^3 + 121x^2 + 51x) \div 3x$

(7) (do it by long division)$(15x^3 + 121x^2 + 51x + 5) \div 3x$

(8) (do it by long division)$(5x^3 + x^2 + 10x + 2) \div 3x$

Excercises 4.5

(1) (do it by writing as a fraction)$51x^7 \div 17x^2$
(2) (do it by writing as a fraction) $51x^7 \div 3x$
(3) (do it by writing as a fraction)$5x^5 \div 2x^2$
(4) (do it by writing as a fraction)$(75x^4 + 121x^3 + 51x^2) \div 3x^2$
(5) (do it by writing as a fraction)$(75x^4 + 121x^3 + 51x^2 + 6x + 1) \div 3x$
(6) (do it by long division)$(75x^4 + 121x^3 + 51x^2 + 6x + 5) \div 3$
(7) (do it by long division)$(75x^4 + 121x^3 + 51x^2 + 6x + 1) \div 3x$
(8) (do it by long division)$(5x^4 + 12x^3 + 5x^2 + x + 1) \div 3x$
(9) (do it by long division)$(75x^4 + 121x^3 + 51x^2 + 6x + 15) \div 3$

4.6. Dividing in $\mathbb{Q}[x]$ II

"a rose is a rose is a rose... Reflexive property"

Gameplan 4.6
(1) *Dividing by Binomial*
(2) *Long Division*

Dividing by binomial

Last section we divided polynomials by *monomials* exclusively. We now tackle the general problem of dividing by a *binomial* or any polynomial. Consider dividing using long division $2 \div 5$. If you think about it for a second you will realize the quotient is 0 with a remainder of 2. This will usually happen when trying to divide an small integer by a larger one. The same holds for polynomials. You will not get very far if you are trying to divide polynomial of small degree by a polynomial of large degree. The quotient will be 0 with remainder equal to the remainder being the original polynomial. For example, $(5x+1) \div x^3 = 0$ with remainder $5x+1$ OR $\frac{5x+1}{x^3}$. Therefore, we will practice our diving skill mostly with cases where we divide a polynomial of large degree by one of smaller degree. The Long Division method will be the primary tool.

Examples: Long division; polynomial by polynomial

(1) Divide $(3x^3 + 5x^2 - 6x - 2) \div (x - 1)$

Solution:

Most steps are identical. One small difference is that we always concentrate on the leading terms, momentarily ignoring the other term. For example, we just concentrate on the leading term $3x^3$ inside and x, the leading term inside. We calculate by inspection $\frac{3x^3}{x} = 3x^2$ to obtain our first term of the quotient.

$$[s \quad 1tage = 2]3x^3 + 5x^2 - 6x - 2x - 1$$

$$1\overline{)\,1\,}$$
$$-1$$
$$0$$

Now, we multiply the $3x^2$ by the entire divisor $(x - 1)$, thus we have to use DL, to get $3x^2(x - 1) = 3x^3 - 3x^2$. This is the quantity we subtract (i.e. change the sign). We obtain...

$$[s \quad \underline{\quad 1}tage = 3]3x^3 + 5x^2 - 6x - 2x - 1$$
$$\begin{array}{r} 1\overline{)\;1} \\ \underline{-\,1} \\ 0 \end{array}$$

Since we have changed the signs already (to subtract), we can simply add the terms and bring down the next term to obtain,

$$[s \quad \underline{\quad 1}tage = 4]3x^3 + 5x^2 - 6x - 2x - 1$$
$$\begin{array}{r} 1\overline{)\;1} \\ \underline{-\,1} \\ 0 \end{array}$$

We now calculate the quotient for leading terms $\frac{8x^2}{x} = 8x$ to get the next term of the quotient...

$$[s \quad \underline{\quad 1}tage = 5]3x^3 + 5x^2 - 6x - 2x - 1$$
$$\begin{array}{r} 1\overline{)\;1} \\ \underline{-\,1} \\ 0 \end{array}$$

We now multiply $8x(x-1) = 8x^2 - 8x$ and subtract, OR change sign and add to obtain....

$$[s \quad \underline{\quad 1}tage = 7]3x^3 + 5x^2 - 6x - 2x - 1$$
$$\begin{array}{r} 1\overline{)\;1} \\ \underline{-\,1} \\ 0 \end{array}$$

Finally, we calculate $\frac{2x}{x} = 2$ for the last part of the quotient...

$$[s \quad \underline{\quad 1}tage = 8]3x^3 + 5x^2 - 6x - 2x - 1$$
$$\begin{array}{r} 1\overline{)\;1} \\ \underline{-\,1} \\ 0 \end{array}$$

Then we have

$$[s \quad \underline{\quad 1}tage = 10]3x^3 + 5x^2 - 6x - 2x - 1$$
$$\begin{array}{r} 1\overline{)\;1} \\ \underline{-\,1} \\ 0 \end{array}$$

Therefore, by [LD] we have $(3x^3 + 5x^2 - 6x - 2) \div (x - 1) = 3x^2 + 8x + 2$

(2) Divide $(10x^4 + 13x^3 + 8x^2 + 8x + 3) \div (2x + 1)$

Solution:

$$
\begin{array}{r|l}
 & \quad\quad\quad 5x^3 + 4x^2 + 2x + 3 \\
\hline
2x+1 & 10x^4 + 13x^3 + 8x^2 + 8x + 3 \\
 & -10x^4 - 5x^3 \\
 & \qquad\quad 8x^3 + 8x^2 \\
 & \qquad -8x^3 - 4x^2 \\
 & \qquad\qquad\quad 4x^2 + 8x \\
 & \qquad\qquad -4x^2 - 2x \\
 & \qquad\qquad\qquad\quad 6x + 3 \\
 & \qquad\qquad\qquad -6x - 3 \\
 & \qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

(3) Divide $(4x^3 +^- 4x^2 +^- 23x + 22) \div (2x - 5)$

Solution:

$$
\begin{array}{r|l}
 & \quad\quad 2x^2 + 3x - 4 \\
\hline
2x-5 & 4x^3 - 4x^2 - 23x + 22 \\
 & -4x^3 + 10x^2 \\
 & \qquad 6x^2 - 23x \\
 & \quad -6x^2 + 15x \\
 & \qquad\qquad -8x + 22 \\
 & \qquad\qquad\quad 8x - 20 \\
 & \qquad\qquad\qquad\quad 2
\end{array}
$$

thus by [LD] $(4x^3 +^- 4x^2 +^- 23x + 22) \div (2x - 5) = 2x^2 + 3x - 4 + \frac{2}{2x-5}$

(4) Divide $(4x^4 + 8x^3 + -3x^2 - x - 4) \div (2x^2 + 3x - 4)$

Solution:

$$
\begin{array}{r|l}
 & \quad\quad\quad 2x^2 + 3x - 4 \\
\hline
2x^2+x+1 & 4x^4 + 8x^3 - 3x^2 - x - 4 \\
 & -4x^4 - 2x^3 - 2x^2 \\
 & \qquad 6x^3 - 5x^2 - x \\
 & \quad -6x^3 - 3x^2 - 3x \\
 & \qquad\qquad -8x^2 - 4x - 4 \\
 & \qquad\qquad\quad 8x^2 + 4x + 4 \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

thus by [LD] $(4x^3 +^- 4x^2 +^- 23x + 22) \div (2x - 5) = 2x^2 + 3x - 4$

(5) Divide $(4x^4 + 8x^3 + -3x^2 - x - 4) \div (3x + 2)$

Solution:

$$
\begin{array}{r|rrrrr}
 & & \frac{4}{3}x^3 & +\frac{16}{9}x^2 & -\frac{59}{27}x & +\frac{91}{81} \\
\hline
3x+2 & 4x^4 & +8x^3 & -3x^2 & -x & -4 \\
 & -4x^4 & -\frac{8}{3}x^3 & & & \\
\cline{2-3}
 & & \frac{16}{3}x^3 & -3x^2 & & \\
 & & -\frac{16}{3}x^3 & -\frac{32}{9}x^2 & & \\
\cline{3-4}
 & & & -\frac{59}{9}x^2 & -x & \\
 & & & \frac{59}{9}x^2 & +\frac{118}{27}x & \\
\cline{4-5}
 & & & & \frac{91}{27}x & -4 \\
 & & & & -\frac{91}{27}x & -\frac{182}{81} \\
\cline{5-6}
 & & & & & -\frac{506}{81}
\end{array}
$$

Excercises 4.6

(1) $(6x^3 + 11x^2 +^- 19x + 10) \div (x - 2)$
(2) $(6x^3 + 11x^2 +^- 19x + 10) \div (x + 3)$
(3) $(6x^3 + 11x^2 +^- 6x + 1) \div (3x + 1)$
(4) $(6x^3 + 11x^2 +^- 19x + 10) \div (3x - 2)$
(5) $(6x^3 + 11x^2 +^- 19x + 10) \div (2x - 3)$
(6) $(x^3 - 1) \div (x - 1)$
(7) $(x^4 - 1) \div (x - 1)$
(8) $(x^5 - 1) \div (x - 1)$

Classroom Exercises

(1) $(3x^3 - 5x^2 + 4x - 2) \div (x + 1)$

(2) $(3x^3 - 5x^2 + 7x - 2) \div (x + 1)$

(3) $(8x^3 - 4x^2 + 5x + 7) \div (2x - 1)$

4.7. Factoring by Distributive Law

"a rose is a rose is a rose... Reflexive property"

Gameplan 4.7
(1) *Factor by DL*
(2) *Practice*

MAIN IDEA

Recall 'To factor' means to break up into multiples. The main tool here will be the distributive law. To *factor completely* means to factor, factor until you can factor no more. Another name for factoring completely is prime factorization. In this section, we will limit our focus to factoring only when we can use the distributive law. Observe...

EXAMPLES

(1) $6x + 4$

Solution:

$$\begin{aligned} 6x + 4 &= 2 \cdot 3x + 2 \cdot 2 && \text{TT} \\ &= 2(3x + 2) && \text{DL} \end{aligned}$$

(2) $16x^2 + 40x + 24$

Solution:

$$\begin{aligned} 16x^2 + 40x + 24 &= 8 \cdot 2x^2 + 8 \cdot 5x + 8 \cdot 3 && \text{TT} \\ &= 8(2x^2 + 5x + 3) && \text{DL} \end{aligned}$$

(3) $16x^5 + 40x^4 + 24x^3$

Solution:

$$\begin{aligned} 16x^5 + 40x^4 + 24x^3 &= 8x^3 \cdot 2x^2 + 8x^3 \cdot 5x + 8x^3 \cdot 3 && \text{B.I.} \\ &= 8x^3(2x^2 + 5x + 3) && \text{DL} \end{aligned}$$

(4) $(blah)x^3 + (blah)2x$

Solution:

$$\begin{aligned} (blah)x^3 + (blah)2x &= (blah)(x^3 + 2x) && \text{DL} \\ &= (blah)(x^2x + 2x) && \text{BI} \\ &= (blah)(x^2 + 2)x && \text{DL} \end{aligned}$$

(5) $(x + 5)x^3 + (x + 5)2x$

Solution:

$$\begin{aligned} (x + 5)x^3 + (x + 5)2x &= (x + 5)(x^3 + 2x) && \text{DL} \\ &= (x + 5)(x^2 \cdot x + 2x) && \text{JAE} \\ &= (x + 5)(x^2 + 2)x && \text{DL} \end{aligned}$$

Excercises 4.7

(1) $10x^3 + 20x^5 - 40x^2 + 100$
(2) $12x^3 + 24x^2 + 16$
(3) $10x + 14x^3 - 35x^2$
(4) $(Z)2 + (Z)x$
(5) $(R)2 + (R)x$
(6) $(\clubsuit)2 + (\clubsuit)x$
(7) $(stuff)2 + (stuff)x$
(8) $(x + 1)2 + (x + 1)x$
(9) $(2x + 1)2 + (2x + 1)x$
(10) $(2x + 1)2 + (2x + 1)x^2 + (2x + 1) \cdot^{-} 3x^5$
(11) $(x^2 + 1)2x + (x^2 + 1)x^2$
(12) $(x + 3)2 + (x + 3)y$
(13) $(x + 3)2 + (x + 3)x + 7(x + 3)y$
(14) $(x + 7)2 + (x + 7)y^2$
(15) $3x^5 + 2x^7$
(16) $3x^5y^{10} + 2x^7y^5$
(17) $(x + 3) + 7(x + 3)y + (x + 3)z^3$
(18) $(x^2 + 3x + 5) + (x^2 + 3x + 5)x^3$

(19) $5(x^2 + 3x + 5) + 3(x^2 + 3x + 5)x + (x^2 + 3x + 5)x^2$

Classroom Exercises: Factor!

(1) $10x^3 + 15x^5 + 5x^8$

(2) $10x^3 + 15x^5 + 25x^2$

(3) $10x^{100} + 3x^{90}$

(4) $10x^{23} + 15x^{23} + 25x^{24}$

(5) $10x^{\frac{1}{2}} + 15x^{\frac{1}{2}} + 25x^2$

(6) $10Y + 3xY$

(7) $10(Z) + 3x(Z)$

(8) $10(\spadesuit) + 2x(\spadesuit)$

(9) $10(2 + x) + 3x(2 + x)$

(10) $10(2 + x) + 15x(2 + x) + 5x^2(2 + x)$

4.8. Factoring by Grouping

"a rose is a rose is a rose... Reflexive property"

Gameplan 4.8
(1) *By Grouping*
(2) *Practice*

By Grouping Idea

This is only one of many methods we will learn to factor polynomials. It involves grouping parts of a polynomials, then finding common factors on each of these groups. At the heart of it all, we will find the world famous *distributive law*. While it does not always guarantee the factorization of a polynomial, the fact that it works often merits close attention. In addition, this idea will set the foundation for bigger, stronger and more powerful methods.

Examples: Factoring by grouping

(1) $8 + 8y + 6y^3 + 6y^4$

Solution:

$$\begin{aligned} 8 + 8y + 6y^3 + 6y^4 &= (8 \cdot 1 + 8y) + (6y^3 \cdot 1 + 6y^3 \cdot y) && \text{ALA and MId} \\ &= 8(1 + y) + 6y^3(1 + y) && \text{DL} \\ &= (8 + 6y^3)(1 + y) && \text{DL} \end{aligned}$$

(2) $1 + 3x + x^2 + 3x^3$

Solution:

$$\begin{aligned} 1 + 3x + x^2 + 3x^3 &= (1 + 3x) + (1 \cdot x^2 + 3x \cdot x^2) && \text{ALA and MId} \\ &= (1 + 3x) \cdot 1 + (1 + 3x)x^2 && \text{MId, DL} \\ &= (1 + 3x)(1 + x^2) && \text{DL} \end{aligned}$$

(3) $1 + 3x - x^2 - 3x^3$

Solution:

$$\begin{aligned}
1 + 3x - x^2 - 3x^3 &= (1 + 3x) + ({}^-x^2 +^- 3x^3) && \text{def a-b, ALA} \\
&= (1 \cdot 1 + 3x \cdot 1) + (1 \cdot^- x^2 + 3x \cdot^- x^2) && \text{MId, Def of Expo, CLM} \\
&= (1 + 3x) \cdot 1 + (1 + 3x)({}^-x^2) && \text{DL} \\
&= (1 + 3x)(1 +^- x^2) && \text{DL} \\
&= (1 + 3x)(1 +^- x^2) && \text{DL(to be continued...)}
\end{aligned}$$

Excercises 4.8

(1) $8 + 16x^2 + 4x^3 + 8x^5$
(2) $6 - 6y + 8y^3 - 8y^4$
(3) $8 - 2x^2 + 12x^3 - 3x^5$
(4) $9 + 15y^3 + 6y^6 + y^9$
(5) $8 + 14y - 4y^2 + y^4$
(6) $-6 + 12x - 2x^2 + 4x^3$
(7) $-8 + 12t - 2t^2 + 3t^3$
(8) $2 - 6y^2 - y^3 + 3yx^5$
(9) $4 - 4x^3 - 3x^6 + 12x^8$
(10) $-4 - 4y^2 + 12y^3 + 12y^5$

Classroom Exercises

(1) $2x^2 + 3x + 8x + 12$

(2) $x^2 + 2x - 3x - 6$

(3) $x^3 + x^2 + x + 1$

(4) $3x^3 +^- 6x^2 + 4x +^- 2$

(5) $3x^3 +^- 6x^2 +^- 4x + 2$

(6) $3x^5 +^- 6x^4 +^- 4^3 + 2x^2$

(7) $3x^5 +^- 5x^4 +^- 3x^3 + 5x^2$

(8) $x^3 + 3x^2 + 2x + 8$

4.9. More Factor by Grouping

"a rose is a rose is a rose... Reflexive property"

Gameplan 4.9

(1) *Factor by Grouping, even if you can't*
(2) *The Brilliant Idea*
(3) *Steps*

One Example

Suppose we wanted to factor the trinomial

$$2x^2 +^- 6x + 4.$$

Thus far we have two possible method at our disposal. We have studied factoring by *Distributive Law* and by *factor grouping.* Neither of these methods is helpful in this case. So we look to our audacity and creativity to fabricate a new, fresh idea that will help us factor this polynomial and all others like it.

This new fresh idea is conceived as follows. Imagine that we could factor the polynomial, just imagine... If we could factor it, what would the factors be? Without knowing the the factors, this is the best we can do.

$$2x^2 +^- 6x + 4 = (ax + b)(cx + d)$$

At first, it may seem useless since we don't know anything about a, b, c, and d. However, we continue to explore in search of hints that would tell us something about these unknowns. A big break through is achieved by multiplying the right side of the equation.

$$\begin{aligned} 2x^2 +^- 6x + 4 &= (ax + b)(cx + d) && \text{Imagine} \\ &= acx^2 + bcx + dax + bd && \text{FOIL} \\ &= acx^2 + (bc + ad)x + bd && \text{DL} \\ \textit{thus...} \quad 2x^2 +^- 6x + 4 &= acx^2 + (bc + ad)x + bd && \text{TP} \end{aligned}$$

This tells us a few hints about the numbers a, b, c, and d represent. If the two polynomials are equal, $2x^2 +^- 6x + 4 = acx^2 + (bc + ad)x + bd$, then their quadratic terms must be equal. This means we must have $2x^2 = acx^2$, which in turn, means that these coefficients must be equal, thus $2 = ac$. This is the first break through. There are not very many ways of factoring 2 into two integers. In fact, the only ways to factor 2 are $2 = 1 \cdot 2$ or $2 =^- 1 \cdot^- 2$. Since $2 = ac$ then, both a, c have to be either ± 1 or ± 2. If stretched far enough this sort of reasoning will lead us to the solution. We summarize what we know about these numbers. Comparing the quadratic coefficients lead us to:

$$2x^2 +^- 6x + 4 = acx^2 + (bc + ad)x + bd$$

$$\begin{aligned} 2 &= ac && \text{Comparing quadratic coefficeints} \\ {}^-6 &= bc + ad && \text{Comparing linear coefficeints} \\ 4 &= bd && \text{Comparing constant terms} \end{aligned}$$

At this point, we will try to refine our strategy. While we do know some things about a, b, c, and d, by comparing the coefficients, it may still be difficult to find out exactly what each of these is. Rather than finding out each individual value, we will be content to find out the quantities $'bc'$ and $'ad'$. In fact, if we could find out the numbers $'bc'$ and $'ad'$ we could work our way backward. We could split the middle term

$$^-6 = bc + ad.$$

In doing so, we could rewrite the polynomial as

$$2x^2 +^- 6x + 4 = 2x^2 + bcx + adx + 4$$

which could then be factored by grouping since $2 = ac$ and $4 = bd$, then we could write the polynomials as..

$$\begin{aligned} 2x^2 +^- 6x + 4 &= 2x^2 + bcx + adx + 4 && \text{Split middle term } ^-6 \\ &= acx^2 + bcx + adx + bd && \text{sub} \\ &= cx(ax + b) + d(ax + b) && \text{DL} \\ &= (cx + d)(ax + b) && \text{DL} \end{aligned}$$

At last, we will factor the polynomial, if we can just get to the first step, *split the middle term* $^-6 = bc + ad$. The final and best clue in splitting the middle term is supplied by multiplying $abcd$.

On the one hand, $bc \cdot ad = ac \cdot bd = 2 \cdot 4 = 8$, on the other hand $bc + ad =^- 6$. So we need two numbers bc and ad, *whose product is 8,* and *whose sum is* $^-6$. So we go on looking at all the possible ways of factoring 8.

$$\text{factoring } 8: \qquad 1 \cdot 8, \qquad ^-1 \cdot ^- 8, \qquad 2 \cdot 4, \qquad ^-2 \cdot ^- 4$$

We now seek the factors whose sum is $^-6$. Clearly, $^-2 \cdot ^- 4$ are the only ones that satisfy the profile, thus we single them out, and proceed...

$$\text{factoring } 8: \qquad 1 \cdot 8, \qquad ^-1 \cdot ^- 8, \qquad 2 \cdot 4, \qquad \underbrace{^-2 \cdot ^- 4}$$

Now we are ready to split the middle term $^-6 =^- 2 +^- 4$, observe...

$$\begin{aligned} 2x^2 +^- 6x + 4 &= 2x^2 + (^-2 +^- 4)x + 4 && \text{N+NT} \\ &= 2x^2 +^- 2x +^- 4x + 4 && \text{DL} \\ &= (2x^2 +^- 2x) + (^-4x + 4) && \text{ALA} \\ &= (2xx + 2x \cdot^- 1) + (^-4x +^- 4 \cdot^- 1) && \text{BI} \\ &= 2x(x +^- 1) +^- 4(x +^- 1) && \text{DL} \\ &= (2x +^- 4)(x +^- 1) && \text{DL} \\ &= (2x + 2 \cdot^- 2)(x +^- 1) && \text{BI} \\ &= 2(x +^- 2)(x +^- 1) && \text{DL} \end{aligned}$$

The process will always succeed if we can *split the middle term* into two pieces whose product is the the product of the constant terms times the quadratic term. This method will be called the *splitting the middle term [SP]* method. If a factorization exists in $\mathbb{Q}[x]$ this method will produce it, guaranteed. A few timely examples are in order.

Examples: Factor by splitting the middle term

(1) $6x^2 +^- 7x +^- 5$

Solution:

We first multiply the first and the last coefficients, $6 \cdot^- 5 =^- 30$. The we try to factor $^-30$ into factors whose sum is $^-7$, the middle term. We simply list all the factors...

$$\text{factors of}^-30 :^- 1 \cdot 30,^- 2 \cdot 15, 2 \cdot^- 15,^- 3 \cdot 10, \underbrace{3 \cdot^- 10}\ldots..$$

We stop here because we have found the sought numbers, $^-7 = 3 +^- 10$. We now substitute this, splitting the middle term, to finish by grouping and factoring.

$$\begin{aligned} 6x^2 +^- 7x +^- 5 &= 6x^2 + (3 +^- 10)x +^- 5 && \text{BI} \\ &= 6x^2 + 3x +^- 10x +^- 5 && \text{DL} \\ &= (6x^2 + 3x) + (^-10x +^- 5) && \text{DL} \\ &= (3x2x + 3x \cdot 1) + (^-5 \cdot 2x +^- 5 \cdot 1) && \text{BI} \\ &= 3x(2x + 1) +^- 5(2x + 1) && \text{DL} \\ &= (3x +^- 5)(2x + 1) && \text{DL} \end{aligned}$$

(2) $-2 + x + x^2$

Solution:

Again, we begin by multiplying the first and the last coefficients. The constant coefficient is $^-2$ while the quadratic coefficient is 1, thus their product is $1 \cdot {}^- 2 = {}^- 2$. Once we multiply the first and last coefficients, we find all possible ways to factor $^-2$, there is not many...

$$\text{factoring } {}^-2: \qquad \underbrace{2 \cdot {}^- 1}, \qquad 1 \cdot {}^- 2$$

We seek a product whose sum is the middle coefficient for $-2 + x + x^2$, the middle coefficient is 1 (even if you don't see it.. the middle term is $x = 1 \cdot x$ by [MId].) Clearly, the factors $2 \cdot {}^- 1$ will do the trick, since their sum is 1.

$$\begin{aligned}
{}^-2 + x + x^2 &= {}^- 2 + 1 \cdot x + x^2 && \text{MI} \\
&= {}^- 2 + (2 + {}^- 1) \cdot x + x^2 && \text{BI} \\
&= {}^- 2 + 2x + {}^- 1x + x^2 && \text{DL} \\
&= ({}^-2 \cdot 1 + 2x) + ({}^-1x + x \cdot x) && \text{ALA} \\
&= ({}^-1 + x)2 + ({}^-1 + x)x && \text{DL} \\
&= ({}^-1 + x)(2 + x) && \text{DL}
\end{aligned}$$

(3) $^-4 + {}^- 6y + 4y^2$

Solution:

As usual we begin by multiplying $^-4 \cdot 4 = {}^- 16$, then split $^-16$ into all possible factors, looking for factors whose sum is $^-6$.

$$\text{factoring } {}^-16: {}^- 1 \cdot 16, 1 \cdot {}^- 16, {}^- 2 \cdot 8, \underbrace{{}^-8 \cdot 2} \ldots$$

$$\begin{aligned}
{}^-4 + {}^- 6y + 4y^2 &= {}^- 4 + (-8 + 2)y + 4y^2 && \text{BI} \\
&= {}^- 4 + {}^- 8y + 2y + 4y^2 && \text{DL} \\
&= ({}^-4 + {}^- 8y) + (2y + 4y^2) && \text{ALA} \\
&= ({}^-4 \cdot 1 + {}^- 4 \cdot 2y) + (2y \cdot 1 + 2y \cdot 2y) && \text{BI} \\
&= {}^- 4(1 + 2y) + 2y(1 + 2y) && \text{DL} \\
&= ({}^-4 + 2y)(1 + 2y) && \text{DL}
\end{aligned}$$

(4) $8 + 6x + x^2$

Solution:

We multiply $8 \cdot 1 = 8$, we consider the possible factors...

$$\text{factoring } 8 : 1 \cdot 8, ^{-}1 \cdot ^{-}8, \underbrace{2 \cdot 4}, ^{-}2 \cdot ^{-}4 ...$$

$$\begin{aligned}
8 + 6x + x^2 &= 8 + (2+4)x + x^2 && \text{AT} \\
&= 8 + 2x + 4x + x^2 && \text{DL} \\
&= (8 + 2x) + (4x + x^2) && \text{ALA} \\
&= (2 \cdot 4 + 2x) + (x4 + xx) && \text{BI} \\
&= 2(4 + x) + x(4 + x) && \text{DL} \\
&= (2 + x)(4 + x) && \text{DL}
\end{aligned}$$

Classroom Exercises

(1) $^{-}70 + 3x + x^2$

(2) $30 + 13z + z^2$

(3) $^{-}4 + 5z + 6z^2$

(4) $6 + 9y + 3y^2$

(5) $^{-}8 + 10t +^{-} 2t^2$

(6) $10x^4 + 30x^2 - 40$

Excercises 4.9

(1) $-20 + 8x + x^2$
(2) $30 + 13x + x^2$
(3) $-10 + 9x + x^2$
(4) $-10 + 9x + x^2$
(5) $-70 + 3x + x^2$
(6) $30 + 13x + x^2$
(7) $-4 + x^2$
(8) $-30 + 7x + x^2$
(9) $7 - 8t + t^2$
(10) $15 + 8z + z^2$
(11) $-10 + 9x + x^2$
(12) $50 + 15x + x^2$
(13) $10 + 7t + t^2$
(14) $-8 + 2t + t^2$
(15) $-14 - 5z + z^2$
(16) $-14 - 5x + x^2$
(17) $-70 + 3t + t^2$
(18) $-10 + 3z + z^2$
(19) $21 - 10y + y^2$
(20) $6 - 5x + x^2$
(21) $-16y + 12y^2 + 4y^3$
(22) $-42t + 20t^2 - 2t^3$
(23) $36t - 24t^2 + 4t^3$
(24) $-6 + 6x + 12x^2$
(25) $-4 - 6y + 4y^2$
(26) $-4 + 5z + 6z^2$
(27) $6 + 9y + 3y^2$
(28) $-12y^2 + 3y^3$
(29) $200y + 40y^2 + 2y^3$
(30) $30y - 4y^2 - 2y^3$
(31) $24z + 14z^2 + 2z^3$
(32) $-28t + 18t^2 - 2t^3$
(33) $-60t + 8t^2 + 4t^3$
(34) $-98x + 28x^2 - 2x^3$
(35) $100z + 30z^2 + 2z^3$
(36) $-48t + 4t^2 + 4t^3$
(37) $20x - 8x^2 - x^3$
(38) $60x + 26x^2 + 2x^3$
(39) $12x + 10x^2 + 2x^3$
(40) $-8 + 10t - 2t^2$
(41) $-6 - 5z - z^2$
(42) $12 + 8z - 4z^2$
(43) $6 - 7x + 2x^2$
(44) $8 + 24x + 16x^2$
(45) $9 + 21t + 12t^2$
(46) $8 + 8x + 2x^2$
(47) $8 - 4x - 4x^2$
(48) $2 + x - x^2$
(49) $4 - 8t + 4t^2$
(50) $2 - 8x^3 + 8x^6$
(51) $6 + 5y^3 - 6y^6$
(52) $6 - 4t^3 - 2t^6$
(53) $8 + 14z^3 + 6z^6$
(54) $4 - 2y^3 - 2y^6$
(55) $6 + 8y - 8y^2$
(56) $-4 + 10x + 6x^2$
(57) $-4 + 12z + 16z^2$
(58) $9 + 15x + 4x^2$
(59) $12 + 21x + 9x^2$
(60) $-4 - 7y - 3y^2$
(61) $12 + 16x + 4x^2$
(62) $1 - 9z^{10}$
(63) $4 - 16y^2$
(64) $4 - 4z^{50}$
(65) $4 - 16z^2$
(66) $x^2 - x + 1$
(67) $4 - 4y - 8y^2$
(68) $16 + 20x + 4x^2$
(69) $x^2 - 1$
(70) $2 - 9z + 9z^2$
(71) $-3 - 10x - 8x^2$
(72) $x^2 + 1$
(73) $2 - 5x + 3x^2$
(74) $-4 - 8y - 3y^2$
(75) $2x^2 + 11x - 7$
(76) $-2 - 2t + 4t^2$
(77) $x^2 + x + 1$
(78) $8 + 4t - 4t^2$
(79) $16 + 4x - 2x^2$
(80) $-4 + 13x + 12x^2$
(81) $1 - 9t^6$
(82) $16 - 16x^2$
(83) $9 - 4x^2$
(84) $9 - 9y^2$
(85) $16 + 32t^2 + 16t^4$
(86) $16 + 32x^2 + 16x^4$
(87) $4 + 16x^3 + 16x^6$
(88) $1 - 6x^2 + 9x^4$
(89) $1 - 8^2x + 16x^4$
(90) $9 - 12x^2 + 4x^4$

4.10. Factoring Famous Polynomials

"a rose is a rose is a rose... Reflexive property"

Gameplan 4.10
(1) *Meet the Famous*
(2) *Recognize the Famous*

Meet The Famous

There is a reason why the famous become famous. Once fame is achieved, *famous polynomials* deserve special attention and VIP treatment. We have gathered all the famous players below. Take some time to meet them and introduce yourself. We will prove a couple of these, leaving the others for the reader to verify.

Summary of Famous Polynomials

Just Famous:

(1) $a^2 - b^2 = (a-b)(a+b)$ Difference of two Squares [DS]
(2) $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ Difference of two Cubes [DC]
(3) $a^3 + b^3 = (a+b)(a^2-ab+b^2)$ Sum of two Cubes [SC]
(4) $a^2 + b^2 =$ I don't know.. Don't try this (yet)

Famous Pascal Polynomials:

(1) $(a+b)^2 = a^2 + 2ab + b^2$ [PP#2]
(2) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ [PP#3]
(3) $(a+b)^4 = a^4 + 4a^3b + 6a^2b^3 + 4ab^4 + b^4$ [PP#4]
(4) $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ [PP#5]

Famous Geometric Series Polynomials:

(1) $x^2 - 1 = (x-1)(x+1)$ [GS#2]
(2) $x^3 - 1 = (x-1)(x^2+x+1)$ [GS#3]
(3) $x^4 - 1 = (x-1)(x^3+x^2+x+1)$ [GS#4]
(4) $x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1)$ [GS#5]

Ask Why

Take the first one for example, *the difference of squares [DS].* It says that every time we have the difference of two squares we can factor as follows.

$$a^2 - b^2 = (a - b)(a + b)$$

Solution:

proof:

$$\begin{aligned}
(a-b)(a+b) &= (a +^- b)(a+b) && \text{def a-b}\\
&= aa + ab +^- ab +^- bb && \text{FOIL}\\
&= aa + (ab +^- ab) +^- bb && \text{ALA}\\
&= aa + 0 +^- bb && \text{AInv}\\
&= aa +^- bb && \text{AId}\\
&= aa - bb && \text{Def a-b}\\
&= a^2 - b^2 && \text{+Expo}
\end{aligned}$$

Take a moment to consider the consequences. If we read it one way, DS says any product of the form $(x-y)(x+y)$ will always have the same form, namely $(x-y)(x+y) = x^2 - y^2$. On the other hand, if we read it right to left, it says every polynomial of the form $x^2 - y^2$ can always be factored the same way, namely $x^2 - y^2 = (x-y)(x+y)$. Like the *difference of squares [DC],* each of the other famous polynomials tells its own story. Consider for example the *Pascal polynomial #3 [PP3].* It says

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Again, we can read it from left to right or from right to left. From left to right it tells you how to multiply any binomial $(x+y)^3$, and from left to right it tells you how to factor any polynomial of the form $x^3 + 3x^2y + 3xy^2 + y^3$. More importantly, we take a moment to think about why it is...

proof of PP3

$$
\begin{aligned}
(x+y)^3 &= (x+y)(x+y)(x+y) && \text{+Expo}\\
&= ((x+y)(x+y))\,(x+y) && \text{ALM}\\
&= \left(x^2+xy+xy+y^2\right)(x+y) && \text{FOIL}\\
&= \left(x^2+2xy+y^2\right)(x+y) && \text{BI}\\
&= x^2(x+y)+2xy(x+y)+y^2(x+y) && \text{DL}\\
&= x^2x+x^2y+2xyx+2xyy+y^2x+y^2y && \text{DL}\\
&= x^3+x^2y+2x^2y+2xy^2+y^2x+y^3 && \text{BI}\\
&= x^3+1x^2y+2x^2y+2xy^2+1y^2x+y^3 && \text{Mid}\\
&= x^3+(1+2)x^2y+(2+1)y^2x+y^3 && \text{DL}\\
&= x^3+3x^2y+3y^2x+y^3 && \text{DL}
\end{aligned}
$$

Recognize the Famous

One way to factor polynomials is to recognize when you come across a famous one. Consider the task of factoring $x^2+2yx+y^2$. As soon as you see this you should recognize it is a famous polynomial [PP2], with a famous factorization, namely;

$$x^2+2yx+y^2=(x+y)(x+y) \qquad [PP2]$$

Of course, these may not always look exactly like this one, but the idea prevails. Consider the following polynomials, and factor.

(1) Factor x^2+6x+9

Solution:

This polynomial needs a little rewriting before we can see it is exactly a PP2, $x^2+2yx+y^2=(x+y)^2$. Observe...

$$
\begin{aligned}
x^2+6x+9 &= x^2+2\cdot 3x+3\cdot 3 && \text{TT}\\
&= x^2+2\cdot 3x+3^2 && \text{+Expo}\\
&= (x+3)^2 && \text{PP2}
\end{aligned}
$$

(2) Factor $x^2+8x+16$

Solution:

$$\begin{aligned} x^2+8x+16 &= x^2+2\cdot 4x+4^2 && \text{BI} \\ &= (x+4)^2 && \text{PP2} \end{aligned}$$

(3) Factor $9x^2+30x+25$

Solution:

$$\begin{aligned} 9x^2+30x+25 &= 3x3x+2\cdot 5\cdot 3x+5\cdot 5 && \text{BI} \\ &= (3x)^2+2\cdot 5(3x)+5^2 && \text{+Expo} \\ &= (3x+5)^2 && \text{PP2} \end{aligned}$$

(4) Factor $x^2+5x+\frac{25}{4}$

Solution:

This one may be a little special. At first it may be difficult to recognize it as a PP2, but the x^2, a perfect square, and the 25, another perfect square, give it away as possibly a PP2...

$$\begin{aligned} x^2+5x+\frac{25}{4} &= x^2+1\cdot 5x+\frac{25}{4} && \text{Mid} \\ &= x^2+2\cdot\frac{1}{2}\cdot 5x+\frac{25}{4} && \text{MInv} \\ &= x^2+2\cdot\left(\frac{1}{2}\cdot 5\right)x+\frac{25}{4} && \text{ALM} \\ &= x^2+2\cdot\left(\frac{5}{2}\right)x+\left(\frac{5}{2}\right)^2 && \text{BI} \end{aligned}$$

Note by now it is easy to recognize it's a perfect PP2. Therefore...

$$\begin{aligned} x^2+5x+\frac{25}{4} &= \dots && \text{..continued} \\ &= x^2+2\left(\frac{5}{2}\right)x+\left(\frac{5}{2}\right)^2 && \text{above..} \\ &= \left(x+\frac{5}{2}\right)^2 && \text{PP2} \end{aligned}$$

(5) Factor $x^3 - 27$

Solution:

We have been practicing recognizing PP2, yet they are all important. Here is a look at one that is the *difference of cubes [DC].*

$$\begin{aligned} x^3 - 27 &= x^3 - 3 \cdot 3 \cdot 3 && \text{TT} \\ &= x^3 - 3^3 && + \text{Expo} \\ &= (x-3)(x^2 + 3x + 3^2) && \text{DC} \\ &= (x-3)(x^2 + 3x + 9) && \text{BI} \end{aligned}$$

The Pascal Story

There was once a fellow that made his living making triangles. His most famous creation became known as the 'Pascals' Triangle'.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

The pattern is to add each pair of adjacent numbers to get the number in the next row below. The relevance is revealed by the following pattern.

$$x^0 = 1$$
$$(x+y)^1 = 1x + 1y$$
$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

The coefficients are precisely the numbers from the Pascals Triangle. The variables also exhibit a clear and intent pattern, the number of x's decreases on each term, while the number of $y's$ increases, as you move from left to right. Consider expanding $(x+y)^5$, we first determine the coefficients from the triangle...

$$(x+y)^5 \qquad 1 \qquad 5 \qquad 10 \qquad 10 \qquad 5 \qquad 1$$

Then, we can supply our $x's$ decreasing from left to right starting with x^5 to yield...

$$(x+y)^5 \qquad 1x^5 \qquad 5x^4 \qquad 10x^3 \qquad 10x^2 \qquad 5x \qquad 1$$

Finally, supply the $y's$ increasing from left to right, starting with no $y's$, then one y, then y^2... to obtain:

$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

Excercises 4.10

(1) $x^2 + 2xy + y^2$
(2) $x^3 + y^3$
(3) $8x^3 + 27$
(4) $(2x)^3 + (5)^3$
(5) $8x^3 - 125$
(6) $t^2 + 2ys + s^2$
(7) $\heartsuit^3 + 3\heartsuit^2\$ + 3\heartsuit\$^2 + \$^3$
(8) $x^3 + 9x^2 + 27x + 27$
(9) $27y^3 - 8$
(10) $8x^3 - 27t^3$
(11) $x^2 + 10x + 25$
(12) $x^2 + 6x + 9$
(13) $x^2 + 14x + 49$
(14) $x^2 + 20x + 100$
(15) $x^2 + 3x + \frac{9}{4}$
(16) $x^2 +^- 7x + \frac{49}{4}$
(17) $x^2 + \frac{2}{3}x + \frac{1}{9}$
(18) $x^2 +^- \frac{3}{5}x + \frac{9}{100}$
(19) $16x^2 - 25$
(20) $16x^2 + 25$
(21) $9x^2 - 1$
(22) $9x^2 - 4$
(23) $9x^2 + 12x + 4$

Classroom Exercises

(1) Prove [PP2] $a^2 + 2ab + b^2 = (a + b)^2$

(2) Prove [GS4] $(x - 1)(x^4 + x^2 + x + 1) = x^3 - 1$

(3) Factor $x^2 - 25$

(4) Factor $4x^2 - 9$

(5) Factor $x^2 + 10x + 25$

(6) Factor $x^2 + 7x + \frac{49}{4}$

(7) Factor $8x^3 - 125$

(8) Expand $(x + y)^6$

4.11. Factoring All

"a rose is a rose is a rose... Reflexive property"

Gameplan 4.11
(1) *Factoring BI*
(2) *Summary*
(3) *Practice*

Factoring By Inspection

Consider factoring $x^2 +^- 7x + 6$. The principal tool we have learned is the *splitting the middle term* method. We multiply $1 \cdot 6 = 6$, then consider the factors of 6,

$$6 : 1 \cdot 6, 2 \cdot 3, \underbrace{^-1 \cdot ^- 6}, ^- 2 \cdot ^- 3$$

Then we factor...

$$\begin{aligned} x^2 +^- 7x + 6 &= x^2 + (^-1 +^- 6)x + 6 && \text{BI} \\ &= x^2 +^- 1x +^- 6x + 6 && \text{DL} \\ &= (x^2 +^- 1x) + (^-6x + 6) && \text{ALA} \\ &= (xx + x \cdot^- 1) + (^-6x +^- 6 \cdot^- 1) && \text{BI} \\ &= x(x + \cdot^-1) +^- 6(x +^- 1) && \text{DL} \\ &= (x +^- 6)(x +^- 1) && \text{DL} \end{aligned}$$

This is a good reliable, step by step method that will factor all factorable trinomials in $\mathbb{Q}[x]$. Here we entertain a different idea. We simply imagine the trinomial can be factored. Recall, we can write

$$x^2 +^- 7x + 6 = (ax + b)(cx + d)$$

The first terms on each of the factors is $acx^2 = x^2$ then $ac = 1$, so we make an educated guess $a = 1$ and $c = 1$. In other words, we guess that the first part of the factorization takes the form:

$$x^2 +^- 7x + 6 = (x + b)(x + d)$$

We now try to find b and d. We know for sure that when we multiply bd we get $bd = 6$, the last term. We consider the possibilities for factoring $bd = 6$. It can only be $2 \cdot 3, ^- 2 \cdot ^- 3, 1 \cdot 6, ^- 1 \cdot ^- 6$. Therefore, we consider the choices...

$$x^2 +^- 7x + 6 \quad ? =? \quad (x + 2)(x + 3)$$

We mentally check to see if this factorization works, we foil the right side and get $(x + 2)(x + 3) = x^2 + 5x + 6$ which is not the correct factorization. So the 6 does not break up as $2 \cdot 3$. We go on and try $6 =^- 1 \cdot^- 6$...and check

$$x^2 +^- 7x + 6? =?(x +^- 1)(x +^- 6)$$

Which works perfectly well. This method of factoring trinomials we call *factoring by inspection [BI].* Consider a second example.

Factor

$$x^2 +^- 2x +^- 15$$

Solution:

We can break up the $x^2 = xx$, almost with certainty the first part of the factorization takes the form

$$x^2 +^- 2x +^- 15 = (x + c)(x + d)$$

We then guess on c and d, whose product is $cd =^- 15$. This narrows it down $^-15 =^- 1 \cdot 15$, or $^-15 =^- 3 \cdot 5$ or $3 \cdot^- 5$.

We try the first one $x^2 +^- 2x +^- 15 \quad =^? (x +^- 1)(x + 15)$ which does not work. We try each one until we find one that works, namely $x^2 +^- 2x +^- 15^? =^? (x +^- 5)(x + 3)$

This method is in many ways inferior to the *splitting the middle term [SP]* method. The [SP] method always works in a methodical way on any factorable trinomial in $\mathbb{Q}[x]$, while this method is based more on trial, error, guessing, and lots of experience. Until you accumulate the experience, you are advised to give preference to the [SP] method.

Summary of Factoring

Note that as you go about your everyday math, you will encounter polynomials that need to be factorized. We have learned to factorize polynomials by using *Distributive Law,* by *splitting the middle term,* by recognizing the famous, *grouping*, and *by inspection.* Unfortunately, when these polynomials are encountered, they will not come with instructions on which method should be used to factorize it. The idea here is to set up a comprehensive plan of attack for the instances. We such a plan below, in what we call 'a summary of factoring'.

Summary of Factoring Skills

(1) Try to factor **using DL**, as in
$$4x^3 + 16x^2 + 12x = 4x(x^2 + 4x + 3)$$
(2) Try to recognize **famous polynomials**, as in
$$x^2 + 2 \cdot 3x + 9 = (x + 3)^2$$
(3) Try by **grouping,** as in
$$3x^3 + 6x^2 + 2x + 4 = (3x^3 + 6x^2) + (2x + 4)..etc$$
(4) If it looks easy, try **by inspection** as in
$$x^2 + 5x + 6 = (x + 2)(x + 3)$$
(5) Try by **splitting the middle** term as in
$$x^2 + 5x + 6 = x^2 + (2 + 3)x + 6..etc$$
(6) Try each one of these a couple more times... if none work, maybe the polynomial is prime and can not be factored in $\mathbb{Q}[x]$, as in
$$x^2 + 3x + 3.$$

Classroom Exercises

(1) $8 + 6y^2 - 8y^3 - 6y^5$

(2) $9 - 6t^2 + t^4$

(3) $27 + 81x + 81x^2 + 27x^3$

(4) $4 + 8y^3 + 4y^6$

(5) $4 + 12y + 9y^2$

(6) $9 - 6x + x^2$

(7) $-1 + 6x - 12x^2 + 8x^3$

Excercises 4.11

(1) $4 - 8x^2 + 3x^4$
(2) $12 + t^2 - 6t^4$
(3) $-8 + 18x^3 - 4x^6$
(4) $4 - 6y^3 + 2y^6$
(5) $-8 + 12t^2 + 8t^4$
(6) $16 - 9z^2$
(7) $4 - 4y^2$
(8) $9 - 4x^2$
(9) $4 - 4z^2$
(10) $4 - x^2$
(11) $9 - x^2$
(12) $9 - 4t^2$
(13) $1 - 4t^2$
(14) $-4 - 10t - 6t^2$
(15) $-6 + 3t + 9t^2$
(16) $-4 - 11t - 6t^2$
(17) $6 + 14z + 8z^2$
(18) $-8 - 16z - 8z^2$
(19) $12 + 18x + 6x^2$
(20) $2 + 6x + 4x^2$
(21) $12 + 6t - 6t^2$
(22) $1 - 2x - 3x^2$
(23) $16 - 8y + y^2$
(24) $4 + 16y^2 + 16y^4$
(25) $4 + 8x^3 + 4x^6$
(26) $16 + 32x^2 + 16x^4$
(27) $16 + 24x + 9x^2$
(28) $4 - 8y + 4y^2$
(29) $6 - 6x^2 + 6x^3 - 6x^5$
(30) $8 - 4x^2 + 12x^3 - 6x^5$
(31) $1 + t - 2t^3 - 2t^4$
(32) $-4 + 2x + 6x^2 - 3x^3$
(33) $4 + 6x + 4x^2 + 6x^3$
(34) $8 + 8x + 2x^2$
(35) $8 + 16z^3 + 6z^6$
(36) $1 - 3x - 2x^2 + 6x^3$
(37) $-6 + 6z^4$
(38) $2 - 4z - 6z^3 + 12z^4$
(39) $-8 + 4x^3 + 4x^6$
(40) $12 + 17x^2 + 6x^4$
(41) $-8 + 6y^2 - y^4$
(42) $16 + 28t^2 + 12t^4$
(43) $-6 - 2x^3 + 8x^6$
(44) $16 + 32x^3 + 16x^6$
(45) $8 - 2z^4$
(46) $-8 + 20x^2 - 8x^4$
(47) $-8 + 12x^2 + 8x^4$
(48) $-2 - 5z^3 - 2z^6$
(49) $-6 + 3x^3 + 9x^6$
(50) $2 + 4x^2 + 2x^4$
(51) $-2 + 3x^2 + 9x^4$
(52) $9 + 21x^3 + 12x^6$
(53) $1 - y^2$
(54) $16 - 9t^2$
(55) $4 - 4y^2$
(56) $9 - 4x^2$
(57) $1 - 9x^2$
(58) $16 - 4x^2$
(59) $9 - 9z^2$
(60) $16 - t^2$
(61) $16 - 4y^2$
(62) $16 - 16x^2$
(63) $9 - 9t^2$
(64) $1 - 4z^2$
(65) $16 - 4y^2$
(66) $4 - 16x^2$
(67) $4 - 4t^2$
(68) $9 - 4x^2$
(69) $9 + 12t + 3t^2$
(70) $-3 + 5y - 2y^2$
(71) $-4 - 5y - y^2$
(72) $4 - 4z^2$
(73) $12 + 22z + 8z^2$
(74) $2 - 5z - 3z^2$
(75) $16 + 28y + 12y^2$
(76) $-4 - 2x + 6x^2$
(77) $-6 + 13x - 6x^2$
(78) $2 - 10x + 12x^2$
(79) $-3 - z + 2z^2$
(80) $-28 - 3z + z^2$
(81) $14 - 9t + t^2$
(82) $-8 + 2x + x^2$
(83) $25 + 10x + x^2$
(84) $8 + 6x + x^2$
(85) $-5 + 4t + t^2$
(86) $7 - 8t + t^2$
(87) $10 + 7y + y^2$
(88) $9 - 6x + x^2$
(89) $-4 + x^2$
(90) $2 - 3z + z^2$

(91) $-4+3y+y^2$
(92) $-3+2t+t^2$
(93) $-4+x^2$
(94) $-20+8y+y^2$
(95) $-4+3t+t^2$
(96) $30+13x+x^2$
(97) $21-10z+z^2$
(98) $40+14x+x^2$
(99) $-30+7x+x^2$
(100) $5z-4z^2-z^3$
(101) $16t+12t^2+2t^3$
(102) $42y-27y^2+3y^3$
(103) $8z-12z^2+4z^3$
(104) $9y-12y^2+3y^3$
(105) $16t+16t^2+4t^3$
(106) $12t+2t^2-2t^3$
(107) $-40x+12x^2+4x^3$
(108) $12x-10x^2+2x^3$
(109) $-105x-6x^2+3x^3$
(110) $-12t+8t^2+4t^3$
(111) $-120x+28x^2+4x^3$
(112) $-21t+10t^2-t^3$
(113) $-10x+8x^2+2x^3$
(114) $-12t+2t^2+2t^3$
(115) $-8+8x-8x^2+8x^3$
(116) $-6-2z-6z^3-2z^4$
(117) $12+10t-8t^2$
(118) $-6+3y^2-6y^3+3y^5$
(119) $2+6x^3+4x^6$
(120) $-4+10x^3-6x^6$
(121) $12+6t-4t^2-2t^3$
(122) $9+6t^3-8t^6$
(123) $-3-8z-4z^2$
(124) $8+6y^2-8y^3-6y^5$
(125) $9-6t^2+t^4$
(126) $27+81x+81x^2+27x^3$
(127) $4+8y^3+4y^6$
(128) $4+12y+9y^2$
(129) $9-6x+x^2$
(130) $-1+6x-12x^2+8x^3$

4.12. Chapter Review

"a rose is a rose is a rose... Reflexive property"

Gameplan 4.12
(1)
(2)

Excercises 4.12:

(1) Add: $(-2 + -x + x^3) + (5 + 2x + x^2)$
(2) Add: $(3 + 4x + x^3) + (4 + -2x + x^3)$
(3) Add: $x^3 + 3x$
(4) Add: $x^2 + x$
(5) Multiply: $(4 + -2z)(2 + 2z^2 + 3z^3)$
(6) Multiply: $(a + b)^2$
(7) Multiply: $(a - b)^2$
(8) Divide: $(15x^5 + 50x^3 - 10x + 30) \div 5x$
(9) Divide: $(-4xy + 6x^3y + 4x^4y) \div 2xy$
(10) Divide: $(6x^3 + -x^2 + x - 1) \div (2x - 1)$
(11) Factor: $12 + 6t - 4t^2 - 2t^3$
(12) Factor: $9 + 6t^3 - 8t^6$
(13) Factor: $-3 - 8z - 4z^2$
(14) Factor: $8 + 6y^2 - 8y^3 - 6y^5$
(15) Factor: $9 - 6t^2 + t^4$
(16) Factor: $27 + 81x + 81x^2 + 27x^3$
(17) Factor: $4 + 8y^3 + 4y^6$
(18) Factor: $4 + 12y + 9y^2$
(19) Factor: $9 - 6x + x^2$
(20) Factor: $-1 + 6x - 12x^2 + 8x^3$
(21) Factor: $x^3 + 9x^2 + 27x + 27$
(22) Factor: $27y^3 - 8$
(23) Factor: $8x^3 - 27t^3$
(24) Factor: $9x^2 - 25$
(25) Factor: $x^2 + 25$
(26) Factor: $9x^2 - 1$
(27) Factor completely: $x^6 - 64$
(28) *Factor: $x^3 + 4$

CHAPTER 5

Solving Polynomial Equations

5.1. Quadratic Machinery: Fractional Exponents and Radicals

"until one day... nothing happened"

Gameplan 5.1
(1) *Radicals*
(2) *Fractional Exponents*
(3) *Caution*
(4) *RP=PR*
(5) *RQ=QR*

RADICALS

Often, we refer to the act of raising a number to the second power as '*squaring the number'*. In other words, 9^2 is read '9 squared', and it means $9 \cdot 9 = 81$. Here, we consider the opposite, namely, *the square root of 9.* By 'the square root of 9' we will mean *the positive real number whose square is 9*, which is 3. It is important to note there are two real numbers whose square is 9, 3 and $^-3$, however, by default *the* square root of any positive number a is the positive real number whose square is a. The notation for *the square root of* a is a *radical sign*

$$\sqrt{a} = \text{ the square root of } a$$

The definition of square root is the same as that of a radical. We will simply refer to is as the *definition of radical* or [rad]. We can summarize the definition of [rad] by the following equation.

$$\left(\sqrt{a}\right)^2 = a \qquad [rad]$$

The following are examples of famous square roots...

(1) $\sqrt{1} = 1$
(2) $\sqrt{4} = 2$
(3) $\sqrt{9} = 3$
(4) $\sqrt{16} = 4$
(5) $\sqrt{25} = 5$
(6) $\sqrt{36} = 6$

(7) $\sqrt{49} = 7$

(8) $\sqrt{64} = 8$

The numbers inside the radicals above are called perfect squares, and these are easy to take the square root of. Nonsquare integers also have square roots but they are not as easy to compute, yet it should be easy to approximate them. Consider for example, $\sqrt{10}$, the positive real number whose square it 10. The square root of 9 is 3, while the square root of 16 is 4, therefore we can estimate the square root of 10 is somewhere between 3 and 4. Consider finding the square root of 87. The closest squares around 87 are 81 and 100. The square root of 81 is 9, while the square root of 100 is 10, therefore we can estimate the square root of 87 is somewhere between 9 and 10.

Once we understand square roots, we are in much better position to understand cubic, forth, and other roots. By the *'cubic root of a real number a'* we will mean, 'the real number whose cube it a. The symbol for the cubic root is also a radical with a '3' indicating *cubic root.* By definition a cubic root and a cubic exponents are still antagonist, and as before we have by definition of radicals

$$\left(\sqrt[3]{a}\right)^3 = a \qquad [rad]$$

Below is a short list of famous cubic roots.

(1) $\sqrt[3]{1} = 1$

(2) $\sqrt[3]{-1} = 1$

(3) $\sqrt[3]{8} = 2$

(4) $\sqrt[3]{^-8} =^- 2$

(5) $\sqrt[3]{27} = 3$

(6) $\sqrt[3]{64} = 4$

(7) $\sqrt[3]{125} = 5$

(8) $\sqrt[3]{^-27} =^- 3$

(9) $\sqrt[3]{^-64} =^- 4$

In a similar way we can define the forth, fifth and nth root of a. Notice one major difference between the square root and the cubic root is that cubic root were defined even for negative real numbers, as in $\sqrt[3]{^-8} =^- 2$, while they square roots were only defined for positive real numbers. We will define square roots for negative numbers in the next section. In fact, any even root of any negative real number will have to wait until the next section.

Fractional Exponents

The time has come for us to define fractional exponents. Indeed we will make a dictionary linking fractional exponents with radicals. This dictionary will be very useful because using fractional exponents, under some caution, will allow us to use some of the famous exponents theorems (recall [P2P], [JAE], and [CSi].) The first part of the dictionary defines fractional exponents of the type $\frac{1}{n}$. Consider this the first installment of the definition of fractional exponents [frac-expo].

$$\boxed{a^{\frac{1}{n}} = \sqrt[n]{a}}$$

The next part of the definition of fractional exponents is inspired by the [P2P] theorem. Consider

$$\left(a^{\frac{1}{n}}\right)^m$$

Using [P2P] we can multiply the exponents to obtain

$$\begin{aligned}
\left(a^{\frac{1}{n}}\right)^m &= a^{\frac{1}{n}\cdot m} && \text{P2P} \\
&= a^{\frac{1}{n}\cdot\frac{m}{1}} && \text{OUT} \\
&= a^{\frac{m}{n}} && \text{MAT}
\end{aligned}$$

On the other hand,

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

We substitute this into the above argument to make the following general definition of fractional exponents [frac-expo]. Recall, even roots were defined only for positive real numbers. Therefore, the following definition inherits the same limitation. It holds only when a is positive OR n is odd.

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m \qquad \text{[frac-expo]}$$

PRACTICE [FRAC-EXPO]

(1) Simplify $\sqrt[3]{8}$

Solution:

$$\begin{aligned}
\sqrt[3]{8} &= 8^{\frac{1}{3}} && \text{frac-expo} \\
&= \left(2^3\right)^{\frac{1}{3}} && \text{BI} \\
&= (2)^{\frac{3}{3}} && \text{P2P} \\
&= (2)^1 && \text{JOT} \\
&= 2 && \text{+Expo}
\end{aligned}$$

(2) Simplify $\sqrt[6]{27}$

Solution:

$$\begin{aligned}
\sqrt[6]{27} &= (27)^{\frac{1}{6}} && \text{frac-expo} \\
&= \left(3^3\right)^{\frac{1}{6}} && \text{BI} \\
&= (3)^{\frac{3}{6}} && \text{P2P} \\
&= (3)^{\frac{1}{2}} && \text{BI} \\
&= \sqrt{3} && \text{frac-expo (2 is the default index in the radical)}
\end{aligned}$$

(3) Simplify $\sqrt[6]{x^9}$

Solution:

$$\begin{aligned}
\sqrt[6]{x^9} &= (x)^{\frac{9}{6}} && \text{frac-expo} \\
&= (x)^{\frac{3}{2}} && \text{BI} \\
&= x^1 \cdot x^{\frac{1}{2}} && \text{JAE} \\
&= x\sqrt{x} && \text{frac-expo}
\end{aligned}$$

(4) Simplify $\sqrt[6]{x^{14}}$

Solution:

$$\begin{aligned}
\sqrt[6]{x^{14}} &= (x)^{\frac{14}{6}} && \text{frac-expo} \\
&= (x)^{\frac{7}{3}} && \text{BI} \\
&= x^2 \cdot x^{\frac{1}{3}} && \text{JAE} \\
&= x^2\sqrt[3]{x} && \text{frac-expo}
\end{aligned}$$

(5) Simplify $\sqrt[6]{x^{20}}$

Solution:

$$\begin{aligned}
\sqrt[6]{x^{20}} &= (x)^{\frac{20}{6}} && \text{frac-expo}\\
&= (x)^{\frac{10}{3}} && \text{BI}\\
&= x^3 \cdot x^{\frac{2}{3}} && \text{JAE}\\
&= x^3\sqrt[3]{x^2} && \text{frac-expo}
\end{aligned}$$

(6) Simplify $x^2y^3\sqrt[6]{(x^3y)^2y}$

Solution:

$$\begin{aligned}
x^2y^3\sqrt[6]{(x^3y)^2y} &= x^2y^3\sqrt[6]{x^6y^2y} && \text{P2P}\\
&= x^2y^3\sqrt[6]{x^6y^3} && \text{JAE}\\
&= x^2y^3\left(x^6y^3\right)^{\frac{1}{6}} && \text{frac-expo}\\
&= x^2y^3\left(x^{\frac{6}{6}}y^{\frac{3}{6}}\right) && \text{P2P}\\
&= x^2y^3\left(xy^{\frac{1}{2}}\right) && \text{BI}\\
&= x^2y^3xy^{\frac{1}{2}} && \text{ALM}\\
&= x^3y^3y^{\frac{1}{2}} && \text{CoLM, JAE}\\
&= x^3y^3\sqrt{y} && \text{frac-expo}
\end{aligned}$$

Caution

Consider the following sequence of thoughts...

$$\begin{aligned}
{}^-1 &= ({}^-1)^1 && \text{+Expo}\\
&= ({}^-1)^{\frac{2}{2}} && \text{JOT}\\
&= ({}^-1)^{2\cdot\frac{1}{2}} && \text{def a/b}\\
&= \left(({}^-1)^2\right)^{\frac{1}{2}} && \text{P2P}\\
&= (1)^{\frac{1}{2}} && \text{NotNot}\\
&= \sqrt{1} && \text{Frac Expo}\\
&= 1 && \text{def rad}
\end{aligned}$$

The results demands our attention;

$$^{-}1 = 1$$

The fallacy is in [P2P]. We first met [P2P] when we were dealing with integer exponents. You are invited to revisit the respective section to see what the reasoning is behind [P2P] as we first encountered it. The problem is that although [P2P] works well for whole numbers, it is not safe to assume it works just as well for fractional exponents, in fact, *it does not hold true for all fractional exponents.* In particular, those where the base is negative and the root is even, as in the case above, where the base is $^{-}1$ and the root is the *square* root. Therefore, caution is advised when dealing with a negative base. We will learn more about square roots of negative numbers in the next section.

Observe the following sequence...

$$\begin{aligned}
-1 &= (-1)^1 && \text{+Expo} \\
&= (-1)^{\frac{2}{2}} && \text{JOT} \\
&= \sqrt{(-1)^2} && \text{Frac-Expon} \\
&= \sqrt{1} && \text{NOTNOT} \\
&= 1 && \text{Def Rad}
\end{aligned}$$

Again, note the definition of fractional exponents was adopted *only* for positive bases, or for odd (not even) roots. Therefore incorrect statement is the [frac-expo] statement.

[SP=PS] AND [SQ=QS]

Here we establish a couple new theorems. The first is *The Square Root of the Product is the Product of the Square Root [SP=PS].* In short is says if a and b are two positive real numbers then

SP=PS

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

And similarly for quotients, when a and b are positive real numbers, the *square root of a quotient is the quotient of the square roots [SQ=QS].*

SQ=QS

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Practice

(1) Simplify $\sqrt{50}$

Solution:

$$
\begin{aligned}
\sqrt{50} &= \sqrt{25 \cdot 2} && \text{TT} \\
&= \sqrt{25}\sqrt{2} && \text{SP=PS} \\
&= 5\sqrt{2} && \text{Rad (def of radical)}
\end{aligned}
$$

(2) Simplify $\sqrt{72}$

Solution:

$$
\begin{aligned}
\sqrt{72} &= \sqrt{36 \cdot 2} && \text{TT} \\
&= \sqrt{36}\sqrt{2} && \text{SP=PS} \\
&= 6\sqrt{2} && \text{Rad}
\end{aligned}
$$

(3) Simplify $\sqrt{12}$

Solution:

$$
\begin{aligned}
\sqrt{12} &= \sqrt{4 \cdot 3} && \text{TT} \\
&= \sqrt{4}\sqrt{3} && \text{SP=PS} \\
&= 2\sqrt{3} && \text{Rad}
\end{aligned}
$$

(4) Simplify $\sqrt{\frac{16}{9}}$

Solution:

$$\begin{aligned} \sqrt{\frac{16}{9}} &= \frac{\sqrt{16}}{\sqrt{9}} && \text{SQ=QS} \\ &= \frac{4}{3} && \text{Rad} \end{aligned}$$

(5) Assume a is positive, simplify $\sqrt{a^2}$

Solution:

$$\begin{aligned} \sqrt{a^2} &= \sqrt{a \cdot a} && \text{+Expo} \\ &= \sqrt{a}\sqrt{a} && \text{SP=PS (this step requires that } a \text{ be positive)} \\ &= \left(\sqrt{a}\right)^2 && \text{+Expo} \\ &= a && \text{Rad} \end{aligned}$$

On a final note, the proper thing to do here is to formally introduce a new family of numbers, *the real numbers,* $\mathbb{R}$. In a short, the family of real numbers contains all rational numbers, and all the numbers near rational numbers.

These include all the previous numbers we have met, whole numbers, natural numbers, integer numbers, rational numbers, plus all roots of all positive numbers, such as $\sqrt{3}$, $\sqrt{10}$, plus all sorts of other numbers, as $\pi \approx 3.14$. In fact one way to describe the real numbers is *the numbers that can be approximated by a decimal number.* This definition will have to do for now. There are more rigorous definitions of the real numbers, yet our course may not be enough to take us there. Therefore, the above definition will have to substitute. Also note the family of real number

Excercises 5.1

Simplify assume all variables are positive real numbers.

(1) $\sqrt{40}$

(2) $\sqrt[3]{40}$

(3) $\sqrt{75}$

(4) $\sqrt{32}$

(5) $\sqrt[3]{250}$

(6) $\sqrt{50x^2}$

(7) $\sqrt{x^3y^2}$

(8) $\sqrt[3]{x^{40}}$

(9) $\sqrt{\frac{25}{9}}$

(10) $\sqrt{\frac{50}{9}}$

(11) $x^3\sqrt[3]{x^{20}}$

(12) $\frac{x^{\frac{1}{3}}y^{\frac{-3}{4}}}{x^{\frac{-2}{3}}y^{\frac{-7}{4}}}$

(13) $\frac{x^{\frac{-1}{3}}y^{\frac{-3}{4}}}{x^{\frac{-2}{3}}y^{\frac{7}{4}}}$

(14) $x^{\frac{1}{3}}(x^{\frac{2}{3}}+x^{\frac{5}{3}})$

(15) $x^{\frac{1}{3}}(x^{\frac{3}{2}}+x^{\frac{5}{3}})$

(16) $\frac{x^{\frac{1}{2}}y^{\frac{-3}{2}}}{x^{\frac{-2}{3}}y^{\frac{-7}{4}}}$

Classroom Exercises 5.1

(1) $\sqrt{20}$

(2) $\sqrt[3]{81}$

(3) $\sqrt{\frac{50}{16}}$

(4) $\frac{x^{\frac{3}{5}}y^{\frac{-3}{4}}}{x^{\frac{-7}{5}}y^{\frac{-1}{4}}}$

(5) $x^{\frac{5}{3}}(x^{\frac{4}{3}} + x^{\frac{-2}{3}})$

(6) $\frac{x^{\frac{1}{4}}y^{\frac{-3}{2}}}{x^{\frac{-1}{4}}y^{\frac{-7}{4}}}$

5.2. Quadratic Machinery: Complex Numbers

"until one day... nothing happened"

Gameplan 5.2
(1) *The Number*
(2) *Negative Radicals*
(3) *Complex Numbers,* $\mathbb{C}$
(4) $+, -, \div, \times, \mathbb{C}$
(5) *Conjugates*

The Number That Could...

(anonymous)

The large full moon reflected off the ocean. The rhythmic sound of the waves offered peace in abundance. All worries and concerns were washed away with each zip of the hazelnut flavored warm coffee. The desk and the comfortable chair were perfectly positioned on the porch as to catch a glimpse of every small tiny ship as it sailed away into the pacific. Summer nights are always perfect in California. The soft music coming from inside the room, the fresh air, the ocean and everything else was perfect. It was a grad orchestration of forces with one focused purpose, to inspire. I sat in that soft comfortable chair leaned back and enjoyed the million thoughts dancing inside my head. A blank paper, a pencil and my awesome coffee was all there was on the desk. On the paper the equation

$$x^2 = -1$$

The equation was screaming, enticing, talking trash, challenging me, saying "you can't solve me!"

Hours went by faster than I would have liked. Days past by, weeks and months...There was no real number that would solve the equation. But the forces were greater. The inspiration divine. I would not be stopped.. and one day it happened. There was no real number solution, I had looked on the positive side on the negative side and all numbers between. Resolved to avoid defeat at all costs, *I invented a number.* From my own imagination, I gathered all my might, my courage, and my audacity, and I thought...I will create a number. I will call it i, and I will solve my problem by declaring $i = \sqrt{-1}$. It's my number so I can make it behave however I please, just as the artists paints the clouds at his whim...

This solves the equation

$$x^2 =^- 1$$

and marks the birth of a grand elegant family of numbers called the complex numbers, $\mathbb{C}$. With the complex numbers also came a batch of fresh new ideas. These ideas include the meaning of negative radicals, a new family of numbers to add, multiply and divide, and a whole new world that adds perspective to our previous views.

Said another way, i is defined to be a number whose square it $^-1$, then by definition of i;

$$i^2 =^- 1$$

Therefore, i is a solution to

$$x^2 =^- 1$$

Negative Radicals

With the invention of i we can now make sense of radicals (i.e. square roots) of negative real numbers. Consider the radical $\sqrt{^{-}1}$, the number whose square is $^{-}1$. Recall when we first defined $\sqrt{4}$ we did so as 'the number whose number square is 4.' But there are two such numbers 2 and -2. By default, we declared to radical to mean the *positive* number whose square is 4. We follow a similar logic here, as we are confronted with the same dilemma. If we define $\sqrt{^{-}1}$ as the number whose square is $^{-}1$, we will find there are two possible choices, i and $-i$ (see examples). By convention, we will define negative radical to be, *the positive* i rather than ^{-}i. We summarize below

Negative Radicals [neg rad]

$$\sqrt{^{-}1} = i$$

In fact, the above definition as a two part definition. We establish what we mean by the square root of $^{-}1$ *then* we can define the radical for any negative number, under the same definition;

Negative Radicals [neg rad]

$$\sqrt{^{-}5} = i\sqrt{5}$$

EXAMPLES

(1) Simplify $\sqrt{-4}$

$$\sqrt{-4} = i\sqrt{4} \qquad \text{neg rad}$$
$$\sqrt{-4} = i2 \qquad \text{neg rad}$$

(2) Simplify $\sqrt{-10}$

$$\sqrt{-10} = i\sqrt{10} \qquad \text{neg rad}$$

(3) Simplify $\sqrt{-15}$

$$\sqrt{-15} = i\sqrt{15} \qquad \text{neg rad}$$

(4) Simplify $\sqrt{^{-}x}$

Solution:

Notice, we do not know the value of x. We don't know if x is positive or negative. This means we don't know if ^-x is positive or negative therefore we don't know if the radical $\sqrt{^-x}$ is positive or negative.

We now focus on learning basic arithmetic in the world of complex numbers. The trick is to obey all the axioms and a new one, that $i^2 = -1$. That is all! enjoy...

Arithmetic in $\mathbb{C}$

(1) Adding in $\mathbb{C}$

$$3 + 5i + 2 + 3i$$

Solution:

$$\begin{aligned}
3 + 5i + 2 + 3i &= && \text{given}\\
&= 3 + 2 + 5i + 3i && \text{ALA}\\
&= 3 + 2 + (5 + 3)i && \text{DL}\\
&= 5 + 8i && \text{AT}
\end{aligned}$$

(2) Multiplying in $\mathbb{C}$

$$(3 + 5i)(2 + 3i)$$

Solution:

$$\begin{aligned}
(3 + 5i)(2 + 3i) &= && \text{given}\\
&= 3 \cdot 2 + 5i \cdot 2 + 3 \cdot 3i + 5i \cdot 3i && \text{FOIL}\\
&= 6 + 10i + 9i + 15i^2 && \text{BI}\\
&= 6 + (10 + 9)i + 15i^2 && \text{DL}\\
&= 6 + 19i + 15i^2 && \text{AT}\\
&= 6 + 19i + 15 \cdot -1 && \text{Def of } i\\
&= 6 + 19i + -15 && \text{BI}\\
&= -9 + 19i && \text{BI}
\end{aligned}$$

(3) Multiplying in $\mathbb{C}$

$$(4 +^- 5i)(2 + 3i)$$

Solution:

$$
\begin{aligned}
(4 +^{-} 5i)(2 + 3i) &= && \text{given} \\
&= 4 \cdot 2 +^{-} 5i \cdot 2 + 3 \cdot 3i +^{-} 5i \cdot 3i && \text{FOIL} \\
&= 8 +^{-} 10i + 9i +^{-} 15i^2 && \text{BI} \\
&= 8 + (^{-}10 + 9)i +^{-} 15i^2 && \text{DL} \\
&= 6 +^{-} i +^{-} 15i^2 && \text{BI} \\
&= 6 +^{-} i +^{-} 15 \cdot -1 && \text{Def of } i \\
&= 6 +^{-} i + 15 && \text{NNT} \\
&= 21 - i && \text{BI}
\end{aligned}
$$

(4) Multiplying in $\mathbb{C}$

$$(4 + 3i)(2 + 3i)$$

Solution:

$$
\begin{aligned}
(4 + 3i)(2 + 3i) &= && \text{given} \\
&= 8 + 12i + 6i + 9i^2 && \text{FOIL} \\
&= 8 + 12i + 6i + 9 \cdot^{-} 1 && \text{def } i \\
&= 8 + 12i + 6i +^{-} 9 && \text{BI} \\
&=^{-} 1 + 18i && \text{BI}
\end{aligned}
$$

(5) Multiplying in $\mathbb{C}$

$$i^7$$

Solution:

$$
\begin{aligned}
i^7 &= iiiiiii && \text{+Expo} \\
&= i^2 i^2 i^2 i && \text{+Expo} \\
&= -1 \cdot -1 \cdot -1 \cdot i && \text{Def of } i \\
&= -1 \cdot i && \text{BI} \\
&= -i && \text{MT}
\end{aligned}
$$

CONJUGATES

Every complex number can be written in the form

$$a + bi$$

This is called the 'standard form' of the complex number. In this form, the a is called the *real part* and the b is called the imaginary part, since it is the coefficient of i. It turns out that something very special happens when we multiply a number such as $a + bi$ by a number of the type $a - bi$. Observe...

$$\begin{aligned}
(a+bi)(a-bi) &= a^2 + abi - abi - b^2i^2 && \text{Foil} \\
&= a^2 +^- b^2i^2 && \text{AInv} \\
&= a^2 +^- b^2(^-1) && \text{Def } i \\
&= a^2 + b^2 && \text{BI}
\end{aligned}$$

We can try to see what this looks like using numbers....

$$\begin{aligned}
(3+4i)(3-4i) &= 3^2 + 12i - 12i - 4^2i^2 && \text{Foil} \\
&= 9 + 0 +^- 4^2i^2 && \text{AInv} \\
&= 9 +^- 16(^-1) && \text{Def } i \\
&= 9 + 16 && \text{BI} \\
&= 25 && \text{BI}
\end{aligned}$$

These pairs of numbers are called conjugates. Products of conjugates always turn out to be real numbers with zero imaginary parts. These become instrumental in dividing complex numbers. Observe...

DIVIDING IN $\mathbb{C}$

(1) divide $\frac{2+3i}{3+4i}$

Solution:

The key here is to somehow get rid of the i's in the denominator. The classic way to do this is to multiply numerator and denominator by the conjugate of the denominator.

$$\begin{aligned}
\frac{2+3i}{3+4i} &= \frac{2+3i}{3+4i}\cdot 1 && \text{MId}\\
&= \frac{2+3i}{3+4i}\cdot\frac{3-4i}{3-4i} && \text{JOT}\\
&= \frac{(2+3i)(3-4i)}{(3+4i)(3-4i)} && \text{MAT}\\
&= \frac{6-8i+9i-12i^2}{9+12i+^-12i+^-16i^2} && \text{FOIL}\\
&= \frac{6-8i+9i+12}{9+0+16} && \text{Bi}\\
&= \frac{18+i}{25} && \text{Bi}\\
&= \frac{18}{25}+\frac{1}{25}i && \text{ATT}
\end{aligned}$$

Excercises 5.2

(1) Evaluate/Simplify $\sqrt{^-9}$
(2) Evaluate/Simplify $\sqrt{^-25}$
(3) Evaluate/Simplify $\sqrt{^-20}$
(4) Evaluate/Simplify $\sqrt{^-50}$
(5) Evaluate/Simplify $\sqrt{^-x}$
(6) Evaluate/Simplify $\sqrt{(^-20)^2}$
(7) *Evaluate/Simplify $\sqrt{i}$
(8) Add $3i+3+^-4+^-5i$
(9) Add $3i+9+^-4+^-5i$
(10) Add $3i+3+^-4+^-2i$
(11) Add $i\pi+3+^-4+^-5i$
(12) Add $\frac{1}{2}i+3+^-4+^-5i$
(13) Multiply i^{10}
(14) Multiply i^{20}
(15) Multiply i^{101}
(16) Multiply i^{-5}
(17) Multiply $(3i+3)(^-4+^-5i)$
(18) Multiply $(3i+9)(^-4+^-5i)$
(19) Multiply $(3i+3)(^-4+^-2i)$
(20) Multiply $(i\pi)(3+^-4+^-5i)$
(21) Multiply $(\frac{1}{2}i+3)(^-4+^-5i)$
(22) Divide $5\div(1+i)$
(23) Divide $(5+i)\div(2+3i)$
(24) Divide $(2+2i)\div(2-3i)$
(25) Divide $3\div i$
(26) Divide $5\div i^3$

Classroom Exercises 5.2

(1) simplify $\sqrt{-32}$

(2) Add $13i + 3 +^{-} 14 +^{-} 5i$

(3) Multiply i^{11}

(4) Multiply $(2 + 3i) \div (2 - 3i)$

(5) Multiply $(7i + 3)(3 +^{-} 5i)$

(6) Multiply $(3i\pi)(3i +^{-} 4 +^{-} 5i)$

(7) Divide $(2 + 3i) \div (2 - 3i)$

5.3. Solving Linear Equations

"until one day... nothing happened"

Gameplan 5.3
(1) *What is 'Solve'*
(2) *What is Linear*
(3) *Annihilate The Coefficient*
(4) *Gather The Linear*
(5) *Move x's to One Side*
(6) *Free The x's*
(7) *Have a Gameplan x's*

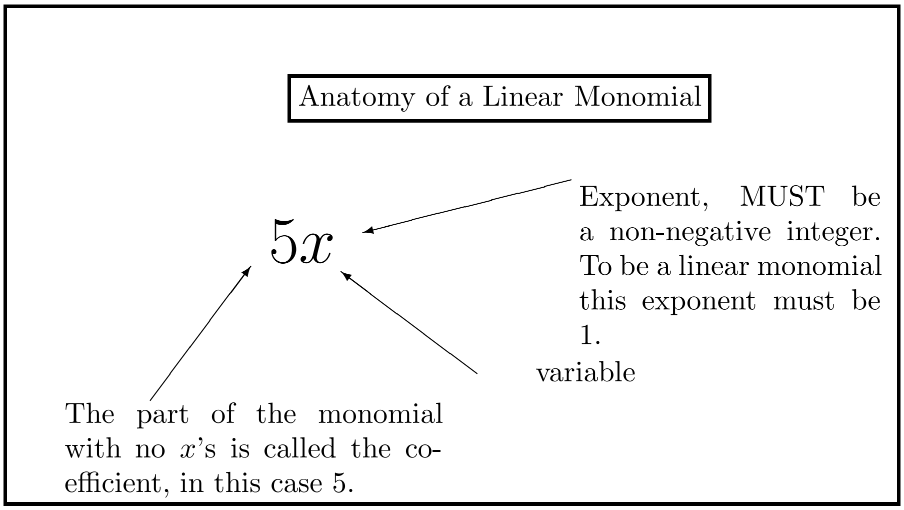
WHAT IS 'SOLVE'

For us, to *solve for* x means use axioms, theorems and definitions to *isolate* x *on one side of the equation, with no* x'*s on the other side.* It's important to emphasize, you are only done 'solving for x' when you have managed to get x one one side, all by itself and no x's on the other side. A couple of examples are in order.

(1) In the following example 'x is solved for'
$$x = \frac{3a+4}{5}$$
(2) In the following example 'x is NOT solved for'
$$x = \frac{3a+4x}{5}$$
(3) In the following example 'y is solved for'
$$y = (3a+4x)5$$
(4) In the following example 'a is NOT solved for'
$$a = \pi(5a+35)$$
(5) In the following example 'a is NOT solved for'
$$5a = \pi(5x+35)$$
(6) In the following example 'x is NOT solved for'
$$x = \frac{-3 \pm \sqrt{9x - 4 \cdot 3}}{2}$$
(7) In the following example 'x is solved for'
$$x = \frac{-3 \pm \sqrt{9 - 4 \cdot 3}}{2}$$

WHAT IS LINEAR

Recall the definition of a monomial.



It is important to note the techniques we discuss here are designed to solve for x in *linear* equations. This means the equation must be made up of linear polynomials in x. We would never use these techniques to solve an equation of the type

$$2x^2 = 5$$

since it is not a linear equation in x.

Annihilate The Coefficient

We will start off with very easy linear equations. In doing so we will accumulate the experience and confidence to tackle the most difficult linear equations of the universe. These type of linear equations have x already on one side and no x's on the other side so they are *almost* solved. The only task left is to annihilate the coefficient so as to leave x completely isolated. The principal tool used here will be [CLM] which says we can multiply both sides by any constant. Observe.

(1) Solve $5x = 10$

Solution:

$$\begin{aligned} 5x &= 10 && \text{given} \\ \frac{1}{5} \cdot 5x &= \frac{1}{5} \cdot 10 && \text{CLM} \\ 1x &= \frac{1}{5} \cdot 10 && \text{MInv} \\ 1x &= \frac{10}{5} && \text{def a/b} \\ x &= 2 && \text{BI} \end{aligned}$$

(2) Solve $5x = 30$

Solution:

$$\begin{aligned} 5x &= 30 && \text{given} \\ \frac{1}{5}\cdot 5x &= \frac{1}{5}\cdot 30 && \text{CLM} \\ 1x &= \frac{1}{5}\cdot 30 && \text{MInv} \\ 1x &= \frac{30}{5} && \text{def a/b} \\ x &= 6 && \text{BI} \end{aligned}$$

(3) Solve $7x = 30$

Solution:

$$\begin{aligned} 7x &= 30 && \text{given} \\ \frac{1}{7}\cdot 7x &= \frac{1}{7}\cdot 30 && \text{CLM} \\ x &= \frac{30}{7} && \text{BI} \end{aligned}$$

(4) Solve $\pi x = 10$

Solution:

$$\begin{aligned} \pi x &= 10 && \text{given} \\ \frac{1}{\pi}\cdot \pi x &= \frac{1}{\pi}\cdot 10 && \text{CLM} \\ x &= \frac{10}{\pi} && \text{BI} \end{aligned}$$

(5) Solve $3x = 10y^3a^2 + 5\sin t + \sqrt{8}$

Solution:

Note, there is nothing to be fear. The right side has no x's. All we need to do is isolate the x on the left side. In other words all we have to do is kill the 3, duck soup!!!

$$3x = 10y^3a^2 + 5\sin t + \sqrt{8} \qquad \text{given}$$

$$\frac{1}{3} \cdot 3x = \frac{1}{3}(10y^3a^2 + 5\sin t + \sqrt{8}) \qquad \text{CLM}$$

$$x = \frac{10y^3a^2 + 5\sin t + \sqrt{8}}{3} \qquad \text{BI}$$

(6) Solve $(3+\pi)x = 8$

Solution:

Note, there is nothing to be fear. The right side has no x's. All we need to do is isolate the x on the left side. In other words all we have to do is kill the $(3+\pi)$, duck soup!!!

$$(3+\pi)x = 8 \qquad \text{given}$$

$$\frac{1}{(3+\pi)} \cdot (3+\pi)x = \frac{1}{3+\pi} \cdot 8 \qquad \text{CLM}$$

$$x = \frac{8}{3+\pi} \qquad \text{BI}$$

(7) Solve $(\clubsuit + \heartsuit)x = 7\diamondsuit + \flat^3$

Solution:

$$(\clubsuit + \heartsuit)x = 7\diamondsuit + \flat^3 \qquad \text{given}$$

$$\frac{1}{\clubsuit + \heartsuit}(\clubsuit + \heartsuit)x = \frac{1}{\clubsuit + \heartsuit}(7\diamondsuit + \flat^3) \qquad \text{CLM}$$

$$x = \frac{7\diamondsuit + \flat^3}{\clubsuit + \heartsuit} \qquad \text{BI}$$

Gather Linear Terms

In this case, we consider the possibility that there may be more than one linear term. The solution is to combine all linear terms using the famous Distributive Law [DL.] Observe

(1) Solve for x in $5x + 8x = 20$

Solution:

$$\begin{aligned} 5x + 8x &= 20 && \text{given} \\ (5+8)x &= 20 && \text{DL} \\ 13x &= 20 && \text{AT} \\ \frac{1}{13} \cdot 13x &= \frac{1}{13} \cdot 20 && \text{CLM} \\ x &= \frac{20}{13} && \text{BI} \end{aligned}$$

(2) Solve for x in $5x + \pi x = 20$

Solution:

$$\begin{aligned} 5x + \pi x &= 20 && \text{given} \\ (5+\pi)x &= 20 && \text{DL} \\ \frac{1}{5+\pi}(5+\pi)x &= \frac{1}{5+\pi} \cdot 20 && \text{CLM} \\ x &= \frac{20}{5+\pi} && \text{BI} \end{aligned}$$

(3) Solve for x in $5x + \pi x + \sqrt{5}x = y^2 + 3$

Solution:

$$\begin{aligned} 5x + \pi x + \sqrt{5}x &= y^2 + 3 && \text{given} \\ (5+\pi+\sqrt{5})x &= y^2 + 3 && \text{DL} \\ \frac{1}{5+\pi+\sqrt{5}}(5+\pi+\sqrt{5})x &= \frac{1}{5+\pi+\sqrt{5}} \cdot (y^2+3) && \text{CLM} \\ x &= \frac{y^2+3}{5+\pi+\sqrt{5}} && \text{BI} \end{aligned}$$

Move all x's to One Side

Here we consider the possibility that not all x's are on one side. Indeed, all previous examples were set up with all terms with x's on one side and all x-less terms on the other side. To get all the x's on one side we will use [CLA]. Similarly, we will get all the x-less terms on the other side.

(1) Solve for x in $5x + 3 = 6x + 7$

Solution:

$$\begin{aligned}
5x + 3 &= 6x + 7 && \text{given}\\
5x + 3 +^{-} 3 &= 6x + 7 +^{-} 3 && \text{CLA}\\
5x &= 6x + 4 && \text{Bi}\\
5x +^{-} 6x &=^{-} 6x + 6x + 4 && \text{CLA}\\
^{-}x &= 4 && \text{BI}\\
(^{-}1)^{-}x &=^{-} 1 \cdot 4 && \text{CLM}\\
x &=^{-} 4 && \text{BI}
\end{aligned}$$

(2) Solve for x in $5x + \pi x = 3 +^{-} \sqrt{5}x$

Solution:

$$\begin{aligned}
5x + \pi x &= 3 +^{-} \sqrt{5}x && \text{given}\\
5x + \pi x + \sqrt{5}x &= 3 +^{-} \sqrt{5}x + \sqrt{5}x && \text{CLA}\\
5x + \pi x + \sqrt{5}x &= 3 && \text{BI}\\
(5 + \pi + \sqrt{5})x &= 3 && \text{DL}\\
\frac{1}{5 + \pi + \sqrt{5}}(5 + \pi + \sqrt{5})x &= \frac{1}{5 + \pi + \sqrt{5}} \cdot (3) && \text{CLM}\\
x &= \frac{3}{5 + \pi + \sqrt{5}} && \text{BI}
\end{aligned}$$

The lesson to learn here is, if x appears on both sides of the equation, we must make sure to use [CLA] to get all x on one side and all x-less stuff on the other side. Yet there is still one obstacle we have not considered. At times, the terms can not be moved from one side to the other because they are trapped inside some parenthesis. Before the moving can take place, the terms must be liberated. These parenthesis can be removed using [DL].

Let the x's Free

Indeed this is the last obstacle to contend with. If any of the x's are inside a set of parenthesis they must be freed before they can be moved from one side to the other side of the equation. Often these parenthesis can be removed using the distributive law [DL]. Once these are removed, terms are free to move from one side of the equation to the other using [CLA].

(1) Solve for x in $5(x + \pi) = 3(2 + x)$

Solution:

$$
\begin{aligned}
5(x+\pi) &= 3(2+x) && \text{given}\\
5x+5\pi &= 6+3x && \text{DL}\\
5x+5\pi+^{-}5\pi+^{-}3x &= 6+3x+^{-}5\pi+^{-}3x && \text{CLA}\\
5x+^{-}3x &= 6+^{-}5\pi && \text{BI}\\
2x &= 6+^{-}5\pi && \text{BI}\\
\frac{1}{2}\cdot 2x &= \frac{1}{2}(6+^{-}5\pi) && \text{CLM}\\
x &= \frac{6+^{-}5\pi}{2} && \text{BI}
\end{aligned}
$$

(2) Solve for x in $5(x+\pi)=\sqrt{3}(2+x)$

Solution:

$$
\begin{aligned}
5(x+\pi) &= \sqrt{3}(2+x) && \text{given}\\
5x+5\pi &= 2\sqrt{3}+\sqrt{3}x && \text{DL}\\
5x+5\pi+^{-}5\pi+^{-}3x &= 2\sqrt{3}+\sqrt{3}x+^{-}5\pi+^{-}\sqrt{3}x && \text{CLA}\\
5x+^{-}\sqrt{3}x &= 2\sqrt{3}+^{-}5\pi && \text{BI}\\
(5+^{-}\sqrt{3})x &= 2\sqrt{3}+^{-}5\pi && \text{DL}\\
\frac{1}{5+^{-}\sqrt{3}}(5+^{-}\sqrt{3})x &= \frac{1}{5+^{-}\sqrt{3}}(2\sqrt{3}+^{-}5\pi) && \text{CLM}\\
x &= \frac{2\sqrt{3}+^{-}5\pi}{5+^{-}\sqrt{3}} && \text{BI}
\end{aligned}
$$

Excercises 5.3Solve for x

(1) $2x+3=5$
(2) $4x=5-x$
(3) $3x+3=5$
(4) $^{-}4x=5-x$
(5) $12x+3=5$
(6) $\pi x=5-x$
(7) $4(2x+2)=2$
(8) $4(2x+2)=2(x+6)$
(9) $4(2x+3)=\pi(x+6)$
(10) $\sqrt{5}(2x+\frac{5}{11})=\pi(x+6)$
(11) $4(2x-2)=5(x+6)$
(12) $4(2x-2)=5(\pi x+6)$
(13) $4(2x-2)=5(2x+6)$
(14) $4(2x-2)=5(\sqrt{5}x+6)$
(15) $4(yx-2)=5(ax+6)$
(16) $y^2(ax-2)=t(rx+z^3)$

(17) $log3(x-2) = \sin 20(x+\frac{1}{2})$
(18) $\clubsuit(3x+\diamondsuit) = 3x+\heartsuit(\heartsuit x1)$

Classroom Exercises 5.3

(1) $3x = 10$

(2) $3x + \sqrt{2}x = 10 + \log 5$

(3) $3x + \sqrt{2}x + \pi x = 10 + \log 5 + 3$

(4) $3(2x - 4) = x + \sqrt{3}(x + 1)$

(5) $3(\heartsuit x - 4) = \diamondsuit x + \sqrt{3}(x + 1) + 10$

(6) $3(\frac{2}{5}x - 4) = \frac{2}{7}x + \sqrt{3}(x + \frac{\pi}{5}) + 10$

5.4. Solving Quadratics: Zero Factor Theorem

"until one day... nothing happened"

Gameplan 5.4
(1) *Zero Factor Theorem [ZFT]*
(2) *ZFT in Action*

Zero Factor Theorem [ZFT]

Suppose I said to you, "I have two numbers in my pocket. I'll call these two numbers a and b. Moreover, when I multiply a and b together I get *zero*. What can we say about these numbers?"... All we know is that $ab = 0$, from this little hint alone we get plenty. In fact, we can conclude that if $ab = 0$ then either $a = 0$ or $b = 0$. This is what we call *the Zero Factor Theorem [ZFT].* Below we prove this excellent theorem.

$ab = 0$	Given
If $a \neq 0$ then the number $\frac{1}{a}$ exists	M.Inv. Axiom
$\frac{1}{a} \cdot ab = \frac{1}{a} \cdot 0$	CLM
$1 \cdot b = 0$	M.Inv. and 0MT
$b = 0$	M.Id

So we can conclude that if $a \neq 0$ then $b = 0$. Similarly, if $b \neq 0$ then $a = 0$. Either way $a = 0$ or $b = 0$. This theorem also applies to product of more than two factors. For example, if $abc = 0$ then we can conclude that $a = 0$ or $b = 0$ or $c = 0$. This amazingly simple idea is the key to solving many higher degree equations. Also note this theorem make use of the very special number 0. It would not work for example if $ab = 6$ (a nor b have to be 6 in this case). That is why it is called the *Zero Factor Theorem*. (Not the six factor theorem).

Examples

(1) Solve for t in

$$(2t + 1)(t + 2) = 0$$

Solution:

$(2t+1)(t+2)=0$			given
$2t+1=0$	or	$t+2=0$	ZFT
$2t={}^-1$	or	$t={}^-2$	CLA
$t=\frac{{}^-1}{2}$	or	$t={}^-2$	CLM

(2) $(2t+1)t=0$

Solution:

$(2t+1)t=0$			given
$2t+1=0$	or	$t=0$	ZFT
$2t={}^-1$	or	$t=0$	CLA
$t=\frac{{}^-1}{2}$	or	$t=0$	CLM

(3) $4t(2t+7)(t+2)=0$

Solution:

		$4t(2t+7)(t+2)=0$			given
$4t=0$	or	$2t+7=0$	or	$t+2=0$	ZFT
$4t=0$	or	$2t=-7$	or	$t=-2$	CLA
$t=\frac{0}{4}$	or	$t=\frac{-7}{2}$	or	$t=-2$	CLM
$t=0$	or	$t=\frac{-7}{2}$	or	$t=-2$	BI

The above examples should make a couple facts clear. One is that the Zero Factor Theorem makes it very easy to solve these equations. The second important fact demanding attention is that ZFT will only work if we can move all terms to one side leaving *zero* on the other side, *and* if we can factor the resulting polynomial. These steps are crucial, and very effective, move all terms to one side and try to factor!

EXAMPLES

(1) $-4+x^2=0$

$-4+x^2=0$			given
$x^2-2^2=0$			B.I.
$(x-2)(x+2)=0$			DS
$x-2=0$	or	$x+2=0$	ZFT
$x=2$	or	$x=-2$	BI, CLA

Notice this degree 2 equation has two solutions... you can easily verify they both satisfy the equation.

(2) $-20+8y+y^2=0$

$-20+8y+y^2=0$			given
$-20+-10y+2y+y^2=0$			BI
$-10(2+y)+y(2+y)=0$			DL
$(-10+y)(2+y)=0$			DL
$-10+y=0$	or	$2+y=0$	ZFT
$y=10$	or	$y=-2$	BI, CLA

Notice this degree 2 equation has two solutions... you can easily verify they both satisfy the equation.

Excercises 5.4

(1) $(2t+7)(t+2)=0$
(2) $8t(2t+7)(t+2)=0$
(3) $8t^2(2t+7)(t+2)=0$
(4) $(2t+1)(t-2)=0$
(5) $(10)(3t+1)(t-7)(t+2)=0$
(6) $(3s+4)s=0$
(7) $(2t+7)(t+2)=0$
(8) $(3s+4)(s+5)s=0$
(9) $(3s-4)(s+5)s=0$
(10) $(3s-4)+(s+5)s=0$
(11) $*(s+4)(5s+y)=1$
(12) $s^3(3s-4)(5s+y)=0$
(13) $s(3s-4)(5s+y+\pi)=0$
(14) $s(3s-\pi)(x^2s+y)=0$
(15) $s(3s-5)(4s+3)(s-1)=0$
(16) $**(s-3)(s^3+s+1)=0$
(17) $-4+3t+t^2=0$
(18) $30+13x+x^2=0$
(19) $21-10z+z^2=0$
(20) $40+14x+x^2=0$
(21) $-30+7x+x^2=0$
(22) $4+12y+9y^2=0$
(23) $9-6x+x^2=0$
(24) $12+6t-4t^2-2t^3=0$
(25) $*-3+8z-4z^2=0$
(26) $*8+6y^2-8y^3-6y^5=0$

Classroom Exercises 5.4

(1) Solve $(s+4)(5s+3)=0$

(2) Solve $x^2(3x-4)(5x+1)=0$

(3) Solve $4(3x-4)(5x+\pi)=0$

(4) $-4+3t+t^2=0$

(5) $30+13x+x^2=0$

(6) $21-10z+z^2=0$

5.5. Solving Quadratics: Square Root Property

"until one day... nothing happened"

Gameplan 5.5
(1) *The Square Root Property [SRP]*
(2) *Practice*

The Square Root Property [SRP]

Here, we concentrate on solving one particular type of equation. The type that look like $x^2 = 4$ or $y^2 = 10$ or $(blah)^2 = a$ The principal tool will be the *square root property [ZRP],* which says...

Suppose than .. $x^2 = a$			given
$x^2 + -a = a + -a$			CLA
$x^2 + -a = 0$			A.Inv.
$x^2 + -(\sqrt{a})^2 = 0$			Def of Rad
$(x - \sqrt{a})(x + \sqrt{a}) = 0$			DS
$x - \sqrt{a} = 0$	or	$x + \sqrt{a} = 0$	ZFT
$x - \sqrt{a} + \sqrt{a} = 0 + \sqrt{a}$	or	$x + \sqrt{a} + -\sqrt{a} = 0 + -\sqrt{a}$	CLA
$x = \sqrt{a}$	or	$x = -\sqrt{a}$	B.I.

This is called *The Square Root Property [SRP]* we can summarize it as:

The Square Root Property [SRP]

If $x^2 = a$ then $x = \pm\sqrt{a}$

Practice [SRP]

(1) Solve $x^2 = 4$

$$\begin{array}{rcll} & x^2 = 4 & & \text{given} \\ x = \sqrt{4} & \text{or} & x = -\sqrt{4} & \text{SRP} \\ x = 2 & \text{or} & x = -2 & \text{Def of Rad} \end{array}$$

(2) Solve $x^2 = 10$

$$\begin{array}{rcll} & x^2 = 10 & & \text{given} \\ x = \sqrt{10} & \text{or} & x = -\sqrt{10} & \text{SRP} \end{array}$$

(3) Solve $(2x+3)^2 = 10$

$$\begin{array}{rcll} & (2x+3)^2 = 10 & & \text{given} \\ 2x+3 = \sqrt{10} & \text{or} & 2x+3 = -\sqrt{10} & \text{SRP} \\ 2x+3+-3 = \sqrt{10}+-3 & \text{or} & 2x+3+-3 = -\sqrt{10}+-3 & \text{CLA} \\ 2x = \sqrt{10}+-3 & \text{or} & 2x = -\sqrt{10}+-3 & \text{B.I.} \\ \frac{1}{2}\cdot 2x = \frac{1}{2}\cdot(\sqrt{10}+-3) & \text{or} & \frac{1}{2}\cdot 2x = \frac{1}{2}\cdot(-\sqrt{10}+-3) & \text{CLM} \\ x = \frac{1}{2}\cdot(\sqrt{10}+-3) & \text{or} & x = \frac{1}{2}\cdot(-\sqrt{10}+-3) & \text{B.I.} \end{array}$$

Excercises 5.5

(1) $x^2 = 15$
(2) $(x-1)^2 = 15$
(3) $(x-5)^2 = 15$
(4) $(3x-5)^2 = 15$
(5) $(3x+9)^2 = 15$
(6) $(3x+9)^2 = \frac{2}{3}$
(7) $(3x+4)^2 = t$
(8) $(3x+4)^2 = 5z$
(9) $3(3x+4)^2 = 5z$
(10) $t(3x-4)^2 = \pi$
(11) $\pi^2(3x-4)^2 = 5\sqrt{7}$
(12) $\pi^2(ex+r\pi)^2 = 5\sqrt{7}$
(13) $4(ax+b)^2 = blah$
(14) $(x+\frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$ (Very Famous)

Classroom Exercises 5.5

(1) $(3x-2)^2 = 4$

(2) $(3x+2)^2 = 9$

(3) $(3x+4)^2 = 10$

(4) $(3x+1)^2 = \frac{2}{3}$

(5) $(3x+4)^2 = 5z$

(6) $3(3x+4)^2 = 5z$

5.6. Solving Quadratics: Completing The Square

"until one day... nothing happened"

Gameplan 5.6
(1) *Review [PP2]*
(2) *Solve Perfect Squares*
(3) *Complete PP2*

Without it

Try to solve the equation $x^2 + 6x + 2 = 0$. Note that none of the previous methods will work here. We can't factor it, and it is not a candidate for the square root property. This type of equation presents the need for brand new and fresh idea to get the job done. Fear not!

Review [PP2]

Recall the famous Pascal Polynomial #2 [PP2];

$$x^2 + 2yx + y^2 = (x + y)^2$$

Now the idea is to turn every degree two polynomial into a [PP2]. At first glance the above polynomial, $x^2 + 6x + 2 = 0$, does not resemble [PP2] in any way. Yet, upon close inspection there are some similarities. Comparing $x^2 + 2yx + y^2$ with $x^2 + 6x + 2$ we find the quadratic term match.

$$\underbrace{x^2} + 2yx + y^2 = \underbrace{x^2} + 6x + 2$$

We continue to compare the linear terms...

$$x^2 + \underbrace{2yx} + y^2 = x^2 + \underbrace{6x} + 2$$

While these do not match, we can do something to make them resemble each other. We can factor the $6 = 2 \cdot 3$. We rewrite the above second polynomial so that the linear terms match with $y = 3$ to obtain...

$$x^2 + \underbrace{2yx} + y^2 = x^2 + \underbrace{2 \cdot 3}x + 2$$

Now we need to get the last term to match.

$$x^2 + 2yx + \underbrace{y^2} = x^2 + 2 \cdot 3x + \underbrace{2}$$

Since we set $y = 3$, then $y^2 = 9$, thus we would like the last term to be 9, rather than a 2. If this was the case we would have a perfect [PP2] or *perfect square*. We could then factor it as...

$$x^2 + 2 \cdot 3x + 9^2 = (x + 3)^2 \qquad \text{PP2}$$

This is the idea behind our present technique to solve degree two equations. We will call this idea the *completing the square* method. Alternatively, we could call it the *completing [PP2]* method. This method is powerful enough solve *all* quadratic equations. Observe how the entire process works to solve degree two equations in the entire universe.

(1) Solve $x^2 + 6x + 2 = 0$

$x^2 + 6x + 2 = 0$	given
$x^2 + 6x = ^- 3$	CLA
$x^2 + 2 \cdot 3x = ^- 3$	TT (make linear term match pp2)
$x^2 + 2 \cdot 3x + 3^2 = ^- 3 + 3^2$	CLA (match last term of pp2)
$(x+3)^2 = ^- 3 + 3^2$	PP2
$(x+3)^2 = 6$	BI
$x + 3 = \pm\sqrt{6}$	Def of Expo
$x = ^- 3 \pm \sqrt{6}$	CLA
$x = ^- 3 + \sqrt{6}$ or $x = ^- 3 - \sqrt{6}$	Def $\pm$

(2) Solve $x^2 + 2x + 3 = 0$

$x^2 + 2x + 3 = 0$	given
$x^2 + 2x + 3 + -3 = 0 + -3$	CLA
$x^2 + 2 \cdot 1x = -3$	B.I.
$x^2 + 2 \cdot 1x + 1 = -3 + 1$	CLA
$x^2 + 2 \cdot 1x + 1 = -2$	BI
$(x+1)^2 = -2$	PP2
$x + 1 = \pm\sqrt{-2}$	SRP
$x = -1 + \sqrt{-2}$ or $x = -1 + -\sqrt{-2}$	CLA
$x = -1 + i\sqrt{2}$ or $x = -1 + -i\sqrt{2}$	B.I. and Def of i

(3) Solve $x^2 + 10x + 3 = 0$

$x^2 + 10x + 3 = 0$			given
$x^2 + 10x + 3 + -3 = 0 + -3$			CLA
$x^2 + 10x = -3$			B.I.
$x^2 + 10x + 25 = -3 + 25$			CLA (the trick!)
$x^2 + 2x \cdot 5 + 5^2 = 22$			B.I.
$(x+5)(x+5) = 22$			PP#2
$(x+5)^2 = 22$			Def of Expo
$x + 5 = \sqrt{22}$	or	$x + 5 = -\sqrt{22}$	ZFT
$x + 5 + -5 = -5 + \sqrt{22}$	or	$x + 5 + -5 = -5 + -\sqrt{22}$	CLA
$x = -5 + \sqrt{22}$	or	$x = -5 + -\sqrt{22}$	B.I.

Note the idea is to move all the terms with x's on one side and add something clever to both sides. What is clever? the square of half of the linear coefficient, exactly what you would expect! nuf' said. The point of the last problem in the last set of exercises was to point out that we will not have a PP2 if the coefficient of the quadratic term is not 1. So at times you may need to clear out the coefficient of the degree 2 term by multiplying both sides of the equation by the inverse of this leading coefficient.

Solve $2t^2 + 3t = 5$

$$
\begin{array}{rclcrcll}
 & & & & 2t^2 + 3t & = & 5 & \text{given} \\
 & & & & \frac{1}{2}(2t^2 + 3t) & = & \frac{1}{2} \cdot 5 & \text{CLM} \\
 & & & & \frac{1}{2} \cdot 2t^2 + \frac{1}{2} \cdot 3t & = & \frac{1}{2} \cdot 5 & \text{D.L.} \\
 & & & & t^2 + \frac{3}{2}t & = & \frac{5}{2} & \text{B.I.} \\
 & & & & t^2 + 2 \cdot \frac{3}{4}t & = & \frac{5}{2} & \text{B.I.} \\
 & & & & t^2 + 2 \cdot \frac{3}{2}t + \left(\frac{3}{4}\right)^2 & = & \frac{5}{2} + \left(\frac{3}{4}\right)^2 & \text{CLA} \\
 & & & & t^2 + 2 \cdot \frac{3}{4}t + \left(\frac{3}{4}\right)^2 & = & \frac{49}{16} & \text{B.I.} \\
 & & & & \left(t + \frac{3}{4}\right)^2 & = & \frac{49}{16} & \text{PP\#2}
\end{array}
$$

$$t + \frac{3}{4} = \sqrt{\frac{49}{16}} \quad \text{or} \quad t + \frac{3}{4} = -\sqrt{\frac{49}{16}} \qquad \text{ZFT}$$

$$t + \frac{3}{4} + -\frac{3}{4} = -\frac{3}{4} + \sqrt{\frac{49}{16}} \quad \text{or} \quad t + \frac{3}{4} - \frac{3}{4} = -\frac{3}{4} + -\sqrt{\frac{49}{16}} \qquad \text{CLA}$$

$$t = -\frac{3}{4} + \frac{7}{4} \quad \text{or} \quad t = -\frac{3}{4} - \frac{7}{4} \qquad \text{B.I.}$$

$$t = 1 \quad \text{or} \quad t = -\frac{10}{4} \qquad \text{B.I.}$$

(1) $t^2 + 8t = 2$
(2) $t^2 + -8t = 2$
(3) $t^2 + 8t = 2t - 6$
(4) $t^2 + 5t + 3 = -t - 6$
(5) $3t^2 + -7t + 3 = -t - 6 + 2t^2$
(6) $t(t + 5) + 2 = 3t - 5$
(7) $t(t + 5) + 2 = 3(-t + 5)$
(8) $t^2 + 3t = -5$
(9) $t^2 + t + 1 = 0$
(10) $t^2 + \frac{3}{5}t = 4$
(11) $t^2 + \pi t = 5$
(12) $t^2 + \sqrt{3}t = 5$
(13) $*t^2 + at + b = 0$ (excellent one)
(14) $*2t^2 + 3t = 5$ (all are excellent)
(15) $3x^2 + 6x = 4$
(16) $2x^2 + 2x + 2 = 0$
(17) $3x^2 + 6x = 4x(x - 5) + 3$
(18) $2x^2 + 6x = 4x(x - 5) + 3$
(19) $3x^2 + 6x = -10$
(20) $3x^2 + 6x = \pi$
(21) $2x(x - 2) = x(3x - 5) + 10$
(22) $3x^2 + \pi x = 4$
(23) $\pi x^2 + 6x = 4$
(24) $*ax^2 + bx + c = 0$
(most famous, official NCSY Problem!)

5.7. Solving Quadratics: Quadratic Formula

"until one day... nothing happened"

Gameplan 5.7
(1) *Where It Comes From*
(2) *What it does*
(3) *Practice*

Where it Comes From

Suppose we were to solve the following equation.

$$ax^2 + bx + c = 0$$

The idea is to solve the equation with generic coefficients a, b, and c. The consequences are of monumental proportions. If we can solve this generic quadratic equation, then we can solve *all* quadratics by simply substituting for the particular coefficients a, b, and c.

Indeed, we will prove that if $ax^2 + bx + c = 0$ then,

$$x = \frac{^-b \pm \sqrt{b^2 +^- 4ac}}{2a}$$

This means we could solve *any* degree 2 equation in one blow. We could solve, for example, $3x^2 + 10x + 7 = 0$, in one line. Namely,

$$x = \frac{-10 \pm \sqrt{10^2 +^- 4 \cdot 3 \cdot 7}}{2 \cdot 3}$$

Which, of course, could be simplified and polished a bit to conclude

$$x = \frac{-10 \pm 4}{6}$$

There is nothing special about this particular quadratic equation. The quadratic formula will solve *any and all* degree two equations, with the greatest of ease. Yet, we take a moment to pay tribute all who ever wonder *why? where did it come from?*

The strategy is to solve by completing the square. The first task is to get rid of the quadratic coefficient, a. We do so by multiplying each side by $\frac{1}{a}$. After we accomplish this, we will need to complete a perfect square (PP2) thus we will need a '$2xy$' as the middle term. We will accomplish this by rewriting $\frac{b}{a}x$ as $2 \cdot \frac{b}{2a}x$ so that $\frac{b}{2a}$ takes the place of 'y'. We will then add a y^2 to both sides to complete the square. Observe....

$$
\begin{aligned}
ax^2 + bx + c &= 0 && \text{given} \\
\frac{1}{a}(ax^2 + bx + c) &= \frac{1}{a} \cdot 0 && \text{CLM} \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 && \text{DL, 0MT} \\
x^2 + \frac{b}{a}x &= \frac{{}^-c}{a} && \text{CLA} \\
x^2 + 2 \cdot \frac{b}{2a}x &= \frac{{}^-c}{a} && \text{BI} \\
x^2 + 2 \cdot \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 &= \frac{{}^-c}{a} + \left(\frac{b}{2a}\right)^2 && \text{CLA} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{{}^-c}{a} + \left(\frac{b}{2a}\right)^2 && \text{PP2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{{}^-c}{a} + \frac{b^2}{4a^2} && \text{BI} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{{}^-c}{a} \cdot \frac{4a}{4a} + \frac{b^2}{4a^2} && \text{Mid,JOT} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{{}^-4ac}{4a^2} + \frac{b^2}{4a^2} && \text{MAT} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 +^- 4ac}{4a^2} && \text{ATT} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 +^- 4ac}{4a^2}} && \text{SRP} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 +^- 4ac}}{\sqrt{4a^2}} && \text{SQ=QS} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 +^- 4ac}}{2a} && \text{BI (is } a \text{ positive?)} \\
x &= \frac{{}^-b}{2a} \pm \frac{\sqrt{b^2 +^- 4ac}}{2a} && \text{CLA} \\
x &= \frac{{}^-b \pm \sqrt{b^2 +^- 4ac}}{2a} && \text{ATT}
\end{aligned}
$$

WHAT IT DOES

(1) $3x^2+5x+1=0$

$$3x^2+5x+1=0 \qquad \text{Given}$$
$$x=\frac{-5\pm\sqrt{5^2-4\cdot 3\cdot 1}}{2\cdot 3} \qquad \text{QF}$$
$$x=\frac{-5\pm\sqrt{25-12}}{6} \qquad \text{BI}$$
$$x=\frac{-5\pm\sqrt{13}}{6} \qquad \text{BI}$$
$$x=\frac{-5+\sqrt{13}}{6} \quad \text{or} \quad x=\frac{-5-\sqrt{13}}{6} \qquad \text{Def } \pm$$

(2) $2x^2+3x+4=0$

$$2x^2+3x+4=0 \qquad \text{Given}$$
$$x=\frac{-3\pm\sqrt{3^2-4\cdot 2\cdot 4}}{2\cdot 2} \qquad \text{QF}$$
$$x=\frac{-3\pm\sqrt{9-16}}{4} \qquad \text{B.I.}$$
$$x=\frac{-3\pm\sqrt{-7}}{4} \qquad \text{B.I.}$$
$$x=\frac{-3\pm i\sqrt{7}}{4} \qquad \text{neg Rad}$$
$$x=\frac{-3+i\sqrt{7}}{4} \quad \text{or} \quad x=\frac{-3-i\sqrt{7}}{4} \qquad \text{def } \pm$$

(3) $\pi x^2+\sqrt{2}x+5=0$

$$\pi x^2+\sqrt{2}x+5=0 \qquad \text{Given}$$
$$x=\frac{-\sqrt{2}\pm\sqrt{(\sqrt{2})^2-4\cdot 5\cdot\pi}}{2\cdot\pi} \qquad \text{QF}$$
$$x=\frac{-\sqrt{2}\pm\sqrt{2-20\pi}}{2\pi} \qquad \text{B.I.}$$

Excercises 5.7

(1) $x^2+x+1=0$
(2) $\clubsuit x^2+\heartsuit x+\spadesuit=0$
(3) $\clubsuit x^2-\heartsuit x+\spadesuit=0$
(4) $2x^2+3x+4=0$
(5) $2(x-3)=2x(3x-4)$
(6) $3x^2+6x=4$
(7) $2x^2+2x+2=0$
(8) $3x^2+6x=4x(x-5)+3$
(9) $2x^2+6x=4x(x-5)+3$
(10) $3x^2+6x=-10$
(11) $3x^2+6x=\pi$
(12) $2x(x-2)=x(3x-5)+10$

(13) $3x^2 + \pi x = 4$
(14) $\pi x^2 + 6x = 4$
(15) Solve for x in $(\sin t)x^2 + (\int t)x + (\log 5) = 0$
(16) Solve for b in $(b \int rdr + \pi b)b + 2b^2 = 3b + eb^2 + 2$
(17) Solve for b in $(b \int rdr + \pi b)(b + \pi) + 2b^2 = 3b + eb^2 + 2$
(18) Solve for h in $\pi h^2 + blah + 1 = 0$
(19) Solve for h in $\pi h^2 + blah + 1 = hat$

Classroom Exercises 5.7

(1) $3x^2 +^- 2x + 4 = 0$

(2) $3x^2 +^- 2x = 5$

(3) $(3x + 4)x = 10$

(4) $\pi x^2 +^- 2x = 5$

(5) $(3x + 4)x = \pi + 3$

(6) $(3x + 4)(x + 1) = 1$

5.8. Solving Higher Degree

"until one day... nothing happened"

Gameplan 5.8
(1) *FTA*
(2) *Degree* $2, 3, 4, 5+$

Fundamental Theorem of Algebra

So the story goes... First proven by a guy they call the *prince of mathematics*, also know as Carl Gauss. In his doctoral thesis of 1799 he presented his first proof of FTA. The proof may be beyond the scope of our book, yet the fruits of this beautiful theorem are not. The theorem says that if we count multiplicity, a degree n polynomials has exactly n roots. In other words, it was not coincidence that the linear equations (degree 1) had exactly 1 root. While the degree 2 equations had exactly 2 roots. Gauss says a deg3 polynomial has 3 solutions, etc... try and see how many of the solutions you can find to each of the polynomials...

Keep in mind that FTA is a statement about *the total number of solutions* of a polynomial equation. It is *not* a theorem describing *how* to solve these equations. Yet, we already have the tools to solve many of these. Let us start with degree one equations.

Degree One Equations

We have the tools to solve any degree one equation. The strategy is to get rid of all parenthesis first, move all terms with x to one side and all the x-less terms to the opposite side, combine the x's and divide by the coefficient. As a brief review consider the following example:

$$2(x+3) = \pi(x+1)$$

Solution:

$$\begin{aligned}
2(x+3) &= \pi(x+1) && \text{Given}\\
2x+6 &= \pi x + \pi && \text{CLA}\\
2x +^- \pi x &= \pi +^- 6 && \text{CLA}\\
(2 +^- \pi)x &= \pi +^- 6 && \text{DL}\\
x &= \frac{\pi +^- 6}{2 +^- \pi} && \text{CLM}
\end{aligned}$$

Degree Two Equations

There is no degree two equation that can stop us. Ultimately, we have the greatest degree-two weapon, *the quadratic formula.* However, you may opt to first check to see if the equation has an obvious factorization so as to use [ZFT]. Otherwise, we pull out the heavy artillery and use [QF]. Note, the manifestation of the *Fundamental Theorem of Algebra [FTA].* These degree-two polynomial equations will have TWO solutions. Observe the following examples:

$$-20 + 8y + y^2 = 0 \text{ (using ZFT)}$$

Solution:

$$\begin{array}{rclr}
-20 + 8y + y^2 = 0 & & & \text{given} \\
(-10 + y)(2 + y) = 0 & & & \text{BI} \\
-10 + y = 0 & \text{or} & 2 + y = 0 & \text{ZFT} \\
y = 10 & \text{or} & y = -2 & \text{BI, CLA}
\end{array}$$

$$3x^2 + 5x + 1 = 0 \text{ (using QF)}$$

$$\begin{array}{rr}
3x^2 + 5x + 1 = 0 & \text{Given} \\
x = \dfrac{-5 \pm \sqrt{5^2 - 4 \cdot 3 \cdot 1}}{2 \cdot 3} & \text{QF} \\
x = \dfrac{-5 \pm \sqrt{25 - 12}}{6} & \text{BI} \\
x = \dfrac{-5 \pm \sqrt{13}}{6} & \text{BI} \\
x = \dfrac{-5 + \sqrt{13}}{6} \quad \text{or} \quad x = \dfrac{-5 - \sqrt{13}}{6} & \text{Def } \pm
\end{array}$$

Higher Degree

For higher degree polynomials, the story does not have such a happy ending. In fact, many years ago, it is said that people held competitions to see who could solve a given degree 3 polynomials. Such competitions were held until one day, some one got the brilliant idea to discover a formula that would resolve all degree 3 polynomial equations.

Tartaglia was famed for his algebraic solution of cubic equations which was published in Cardan's Ars Magna.

Tartaglia's proper name was Niccolo Fontana although he is always known by his nickname. When the French sacked Brescia in 1512 the soldiers killed Tartaglia's father and left him for dead with a sabre wound that cut his jaw and palate. The nickname Tartaglia means the 'stammerer' and one can understand why he stammered.

Tartaglia was self taught in mathematics but having an extraordinary ability was able to earn his living selling sports cards on ebay.

The first person known to have solved cubic equations algebraically was del Ferro. On his deathbed del Ferro passed on the secret to his (rather poor) student Fior. A competition to solve cubic equation was arranged between Fior and Tartaglia. Tartaglia, by winning the competition in 1535, is famed as the discoverer of a formula to solve cubic equations. Because negative numbers were not used there was more than one type of cubic equation and Tartaglia could solve all types, Fior only one type. Tartaglia confided his solution to Cardan on condition that it not be published. The method was, however, published by Cardan in Ars Magna in 1545.

Unfortunately, our journey will fall short of discussing what this 'cubic formula' is. However, we do have some tools that will help us solve *some* degree 3 equations. Specifically, we have the tools to solve those polynomial equations that can be factored by grouping, by using distributive law, or by recognizing as a famous one. Observe the following examples,

(1) Solve $3x^3 + 5x^2 + 2x = 0$

Solution:

We can attempt this one by using Distributive Law first..

$$3x^3 + 5x^2 + 2x = 0 \qquad \text{Given}$$
$$x(3x^2 + 5x + 2) = 0 \qquad \text{DL}$$
$$x(3x + 2)(x + 1) = 0 \qquad \text{BI (or split middle)}$$
$$x = 0 \qquad 3x + 2 = 0 \qquad x + 1 = 0 \qquad \text{ZFT}$$
$$x = 0 \qquad 3x = {}^- 2 \qquad x = {}^- 1 \qquad \text{CLA}$$
$$x = 0 \qquad x = {}^- 2/3 \qquad x = {}^- 1 \qquad \text{CLM}$$

Note we obtain 3 solutions for our degree 3 polynomial.

(2) Solve $3x^3 + 5x^2 + 6x + 10 = 0$

Solution:

We can attempt this one by grouping first..

$$3x^3 + 5x^2 + 6x + 10 = 0 \qquad \text{Given}$$
$$x^2(3x + 5) + 2(3x + 5) = 0 \qquad \text{DL}$$
$$(x^2 + 2)(3x + 5) = 0 \qquad \text{DL}$$
$$x^2 + 2 = 0 \qquad 3x + 5 = 0 \qquad \text{ZFT}$$
$$x^2 = {}^- 2 \qquad 3x = {}^- 5 \qquad \text{CLA}$$
$$x^2 = {}^- 2 \qquad x = {}^- 5/3 \qquad \text{CLM}$$
$$x = \pm\sqrt{{}^-2} \qquad x = {}^- 5/3 \qquad \text{SRP}$$
$$x = \pm i\sqrt{2} \qquad x = {}^- 5/3 \qquad \text{neg rad}$$

Note we obtain 3 solutions for our degree 3 polynomial.

(3) Solve $x^3+6x^2+12x+8=0$

Solution:

We will solve this one by recognizing it is a famous polynomial, a PP3.

$x^3+6x^2+12x+8=0$	Given
$x^3+3x^2 2+3x2^2+2^3=0$	rewrite-BI
$(x+2)^3=0$	PP3
$(x+2)(x+2)(x+2)=0$	+Expo
$x+2=0 \quad x+2=0 \quad x+2=0$	ZFT
$x=-2 \quad x=-2 \quad x=-2$	CLA

Note we obtain 3 solutions (counting multiplicity) for our degree 3 polynomial.

Degree Four Polynomial Equations

The story continues with degree 4 polynomials. Although will not get to see the degree formula to solve *all* degree 3 equations, such a formula was found. The next challenge was to solve degree 4 equations. Such challenge was at last defeated. A degree 4 formula was found, and using it, one can solve *any* degree 4 polynomial. Again, our current treatment will not be enough to access the degree 4 formula. We will be limited to the same sort of options as for degree 3, factor by using [DL], factor by splitting the middle, factor by grouping, or factor by recognizing a famous polynomial. Consider for example the solutions of

$$x^4+4x^2-5=0$$

Solution:

$x^4+4x^2-5=0$	given
$(x^2-1)(x^2+5)=0$	BI or splitt middle
$(x-1)(x+1)(x^2+5)=0$	DS
$x-1=0 \quad x+1=0 \quad x^2+5=0$	ZFT
$x=1 \quad x=-1 \quad x^2=-5$	CLA
$x=1 \quad x=-1 \quad x=\pm\sqrt{-5}$	SRP
$x=1 \quad x=-1 \quad x=\pm i\sqrt{5}$	neg rad

Note we have found 4 solutions for our degree 4 polynomial equation.

Excercises 5.8

(1) $4-8x+3x^2=0$
(2) $x^2+2x+1=0$
(3) $x^2+x+1=0$
(4) $x^2+2x-24=0$
(5) $x^2+2x-15=0$
(6) $x^2+3x-10=0$
(7) $x^2+3x-18=0$
(8) $x^2+3x+18=0$
(9) $x^3=27$
(10) $x^3=-8$
(11) $x^4=16$
(12) $4-8x^2+3x^4=0$
(13) $12+t^2-6t^4=0$
(14) $-8+18x^3-4x^6=0$
(15) $4-6y^3+2y^6=0$
(16) $-8+12t^2+8t^4=0$
(17) $16-9z^2=0$
(18) $4-4y^2=0$
(19) $x^2=1$ (famous)
(20) $x^3=1$ (famous)
(21) $x^4=1$ (famous)
(22) $x^5=1$ (famous)
(23) $x^6=1$ (famous)
(24) $-4-10t-6t^2=0$
(25) $2+6x+4x^2=0$
(26) $4+16y^2+16y^4=0$
(27) $4+8x^3+4x^6=0$
(28) $16+32x^2+16x^4=0$
(29) $16+24x+9x^2=0$
(30) $4-8y+4y^2=0$
(31) $6-6x^2+6x^3-6x^5=0$
(32) $8-4x^2+12x^3-6x^5=0$
(33) $1+t-2t^3-2t^4=0$
(34) $-4+2x+6x^2-3x^3=0$
(35) $4+6x+4x^2+6x^3=0$
(36) $8+8x+2x^2=0$
(37) $8+16z^3+6z^6=0$
(38) $1-3x-2x^2+6x^3=0$
(39) $^*1+3x-5x^2+6x^3=0$
(40) $-6+6z^4=0$
(41) $2-4z-6z^3+12z^4=0$
(42) $-8+4x^3+4x^6=0$
(43) $12+17x^2+6x^4=0$
(44) $-8+6y^2-y^4=0$
(45) $16+28t^2+12t^4=0$
(46) $-6-2x^3+8x^6=0$
(47) $16+32x^3+16x^6=0$
(48) $8-2z^4=0$
(49) $-8+20x^2-8x^4=0$
(50) $-8+12x^2+8x^4=0$
(51) $-2-5z^3-2z^6=0$
(52) $-6+3x^3+9x^6=0$
(53) $2+4x^2+2x^4=0$
(54) $-2+3x^2+9x^4=0$
(55) $9+21x^3+12x^6=0$
(56) $5z-4z^2-z^3=0$
(57) $16t+12t^2+2t^3=0$
(58) $-6-2z-6z^3-2z^4=0$
(59) $-6+3y^2-6y^3+3y^5=0$
(60) $2+6x^3+4x^6=0$
(61) $-4+10x^3-6x^6=0$
(62) $9+6t^3-8t^6=0$
(63) $8+6y^2-8y^3-6y^5=0$
(64) $9-6t^2+t^4=0$
(65) $4+8y^3+4y^6=0$

CHAPTER 6

Miscellaneous Solving

6.1. Rationales

RATIONAL EQUATIONS

"until one day... nothing happened"

Gameplan 6.1
(1) *What are*
(2) *Strategy*
(3) *Practice*

Recall, rational expressions are fractions where the numerator and denominator are both polynomials. The key to solving equations with rational expressions is that if we multiply both sides by some clever polynomial we may kill all the denominators and thus be left with a polynomial equation. Since we have been practicing polynomials equations of various degree these should pose no problem. Note the way to kill all denominators is to multiply both sides of the equation by the LCM of the denominators, and to find the LCM it is often helpful to factor the denominators.

EXAMPLES

(1) Solve $\frac{2}{x} + \frac{3}{x+1} = \frac{7}{x+1}$

Solution:

Note one way to kill the first denominator is to multiply by x, thus we will multiply both sides by x. Meanwhile we would also like to get rid of the $x + 1$ denominators. To accomplish this we can multiply both sides by $x + 1$. Or we kill both denominators at once by multiplying both sides by $x(x + 1)$. The result, in this case, is a routine linear equation. Observe;

$$\begin{aligned}
\frac{2}{x} + \frac{3}{x+1} &= \frac{7}{x+1} && \text{given}\\
x(x+1)\left(\frac{2}{x} + \frac{3}{x+1}\right) &= x(x+1)\left(\frac{7}{x+1}\right) && \text{CLM}\\
x(x+1)\left(\frac{2}{x}\right) + x(x+1)\left(\frac{3}{x+1}\right) &= x(x+1)\left(\frac{7}{x+1}\right) && \text{DL}\\
2(x+1) + 3x &= 7x && \text{BI}\\
2x + 2 + 3x &= 7x && \text{DL}\\
2 &= 2x && \text{BI}\\
1 &= x && \text{CLM}
\end{aligned}$$

(2) Solve $\frac{2x}{12} + \frac{3}{36} = \frac{7}{40}$

Solution:

Note we would like to get rid of all denominators. One way to do this is to multiply each term by 12, then by 36, then by 40. Another way to do it is by finding the LCM between 12, 36, and 40, then multiplying each term by the LCM. This is the preferred way. To find the LCM we will prime factorize each denominator first. Rewriting the equation as

$$\frac{2x}{2^2 \cdot 3} + \frac{3}{2^2 \cdot 3^2} = \frac{7}{2^3 \cdot 5}$$

It is now clear that to kill all denominators we can multiply both sides of the equation by the LCM, $2^3 \cdot 3^2 \cdot 5$.

$$\begin{aligned}
\frac{2x}{2^2 \cdot 3} + \frac{3}{2^2 \cdot 3^2} &= \frac{7}{2^3 \cdot 5} && \text{given}\\
2^3 \cdot 3^2 \cdot 5\left(\frac{2x}{2^2 \cdot 3} + \frac{3}{2^2 \cdot 3^2}\right) &= 2^3 \cdot 3^2 \cdot 5\left(\frac{7}{2^3 \cdot 5}\right) && \text{CLM}\\
2^3 \cdot 3^2 \cdot 5\left(\frac{2x}{2^2 \cdot 3}\right) + 2^3 \cdot 3^2 \cdot 5\left(\frac{3}{2^2 \cdot 3^2}\right) &= 2^3 \cdot 3^2 \cdot 5\left(\frac{7}{2^3 \cdot 5}\right) && \text{DL}\\
2 \cdot 3 \cdot 5 \cdot 2x + 5 \cdot 3 &= 3^2 \cdot 7 && \text{BI}\\
60x + 15 &= 63 && \text{BI}\\
60x &= 48 && \text{CLA}\\
x &= \frac{48}{60} && \text{CLM}\\
x &= \frac{4}{5} && \text{BI}
\end{aligned}$$

(3) Solve $\frac{3}{6x^2-5x-4} + \frac{2}{2x+1} = \frac{7}{5(3x-4)}$

Solution:

This may look like a more difficult problem, but the step needed are exactly the same as the steps used in the previous problem. We begin by factoring the denominators, finding the LCM, multiplying both sides by LCM, and that should kill all fractions. The rest is Duck Soup

$$\begin{aligned}
\frac{3}{6x^2-5x-4}+\frac{2}{2x+1} &= \frac{7}{15x-20} && \text{given}\\
\frac{3}{(2x+1)(3x-4)}+\frac{2}{2x+1} &= \frac{7}{5(3x-4)} && \text{B.I.}\\
5(2x+1)(3x-4)\left(\frac{3}{(2x+1)(3x-4)}+\frac{2}{2x+1}\right) &= 5(2x+1)(3x-4)\left(\frac{7}{5(3x-4)}\right) && \text{C.L.M}\\
5(2x+1)(3x-4)\left(\frac{3}{(2x+1)(3x-4)}\right)+5(2x+1)(3x-4)\left(\frac{2}{2x+1}\right) & \\
&= 5(2x+1)(3x-4)\frac{7}{5(3x-4)} && \text{D.L.}\\
5\cdot 3+5\cdot 2(3x-4) &= (2x+1)\cdot 7 && \text{B.I.}\\
15+10(3x-4) &= 14x+7 && \text{B.I., D.L.}\\
15+30x-40 &= 14x+7 && \text{D.L.}\\
30x-25 &= 14x+7 && \text{B.I.}\\
16x &= 38 && \text{B.I.}\\
x &= 2 && \text{B.I.}
\end{aligned}$$

(4) Solve $\frac{x+1}{x-1}+\frac{x}{x+1}=\frac{2x-1}{x+1}$

Solution:

This time there is nothing to factor, each of the denominators is a prime polynomial in $\mathbb{Q}[x]$. The LCM is the product $(x+1)(x-1)$

$$\begin{aligned}
\frac{x+1}{x-1}+\frac{x}{x+1} &= \frac{2x-1}{x+1} && \text{given}\\
(x+1)(x-1)\left(\frac{x+1}{x-1}+\frac{x}{x+1}\right) &= (x+1)(x-1)\left(\frac{2x-1}{x+1}\right) && \text{CLM}\\
(x+1)(x-1)\left(\frac{x+1}{x-1}\right)+(x+1)(x-1)\left(\frac{x}{x+1}\right) &= (x+1)(x-1)\left(\frac{2x-1}{x+1}\right) && \text{D.L.}\\
(x+1)(x+1)+(x-1)(x) &= (x-1)(2x-1) && \text{B.I.}\\
x^2+2x+1+x^2-x &= 2x^2-3x+1 && \text{Foil, DL}\\
2x^2+x+1 &= 2x^2-3x+1 && \text{B.I.}\\
x &= -3x && \text{B.I.}\\
4x &= 0 && \text{C.L.A}\\
x &= 0 && \text{B.I.}
\end{aligned}$$

EXCERCISES 6.1

(1) Solve $\frac{x}{3}+\frac{2}{5}=7$

(2) Solve $\frac{5}{3} + \frac{2}{x} = 7$
(3) Solve $\frac{^-2}{3} + \frac{2}{x^2} = 7$
(4) Solve $\frac{^-2}{3+x} + \frac{2}{x} = \frac{7}{3x+9}$
(5) Solve $\frac{^-2}{3+x^2} + \frac{2}{x} = \frac{7}{3x^2+9}$
(6) Solve $\frac{2}{x^2-1} + \frac{2}{x-1} = \frac{7}{x+1}$

6.2. Radicals

"until one day... nothing happened"

Gameplan 6.2
1. *What are*
2. *Strategy*
3. *Practice*

EQUATIONS WITH RADICALS

The strategy is to make heavy use of the definition of the *radical.* Recall that by definition, if $a \in \mathbb{R}$ then $\sqrt{a}$ is a number that when we square it, we get a. Thus,

$$(\sqrt{blah})^2 = blah \qquad \text{Definition of Radical}$$

$$\text{also...} \qquad (\sqrt[n]{blah})^n = blah \qquad \text{Definition of Radical}$$

We will also need to use a theorem that says *we can square both sides of an equation.* These two facts will be the foundation of our strategy. We now prove this theorem.

SQUARE BOTH SIDES THEOREM[SBST]

If $a = b$ then $a^2 = b^2$

Proof:

	$a = b$	given
(2)	$a \cdot a = a \cdot b$	CLM
(3)	$a^2 = a \cdot b$	Def of Exponents
	$a = b$	Given
(5)	$a \cdot b = b \cdot b$	CLM
(6)	$a \cdot b = b^2$	Def of Exponent
(7)	$a^2 = b^2$	Transitivity Property on Eqs #3 and #6

You can take this one to the bank!!! It is proven, from now on there should never ever be any doubt in your mind that you may square both sides of an equation. We have proven it! It is a mathematical certainty.

EXAMPLES

(1) Solve $\sqrt{2x+1} = 3$

Solution:

$\sqrt{2x+1} = 3$	Given
$(\sqrt{2x+1})^2 = 3^2$	SBST
$2x+1 = 9$	Def of Rad
$2x = 8$	CLA
$x = 4$	CLM

To the bank!... however, one should realize that the act of squaring both sides of an equation in some sense increases the degrees of the equations increasing the number of solutions and perhaps introducing entrenous solutions... so just to be safe it is recommended that you check the solutions by plugging them back into the original equation.

(2) Solve $\sqrt{2x^2+1} = 3$

Solution:

$\sqrt{2x^2+1} = 3$	Given
$(\sqrt{2x^2+1})^2 = 3^2$	SBST
$2x^2+1 = 9$	Def of Rad
$2x^2 = 8$	CLA
$x^2 = 4$	CLM
$x = \pm\sqrt{4}$	SRP
$x = \pm 2$	Def of Rad

So, we conclude that the solutions are $x = 2$ and/or $x = -2$. After checking both of these we see that they both work so we are done!

(3) $2\sqrt{3x-2} = 5$

Solution:

$2\sqrt{3x-2} = 5$	given
$(2\sqrt{3x-2})^2 = 5^2$	SBST
$(2\sqrt{3x-2})(2\sqrt{3x-2}) = 5 \cdot 5$	Def of Exponents
$2 \cdot 2\sqrt{3x-2} \cdot \sqrt{3x-2} = 5 \cdot 5$	ALM, Comm Law Mult
$4(\sqrt{3x-2})^2 = 25$	Def of Exponents, Times Tables
$4(3x-2) = 25$	Def of Rad
$12x - 8 = 25$	D.L.
$12x = 33$	CLA
$x = \dfrac{33}{12}$	CLM

(4) $\sqrt{4x+1} = \sqrt{x+2} + 1$

Solution:

$\sqrt{4x+1} = \sqrt{x+2} + 1$	given
$(\sqrt{4x+1})^2 = (\sqrt{x+2} + 1)^2$	SBST
$4x + 1 = (\sqrt{x+2} + 1)^2$	Def of Rad
$4x + 1 = (\sqrt{x+2} + 1)(\sqrt{x+2} + 1)$	Def of Expo
$4x + 1 = (\sqrt{x+2})^2 + 2(\sqrt{x+2}) + 1$	Foil
$4x + 1 = x + 2 + 2(\sqrt{x+2}) + 1$	Def of Rad
$4x + 1 = x + 3 + 2(\sqrt{x+2})$	B.I.
$3x - 2 = 2\sqrt{x+2}$	CLA
$(3x-2)^2 = (2\sqrt{x+2})^2$	SBST
$9x^2 - 12x + 4 = 4(x+2)$	Foil on Left, Def of Rad on Right
$9x^2 - 12x + 4 = 4x + 8$	D.L.
$9x^2 - 16x - 4 = 0$	CLA
$(x-2)(9x+2) = 0$	B.I.
$x - 2 = 0 \quad \text{or} \quad 9x + 2 = 0$	ZFT
$x = 2 \quad \text{or} \quad x = \dfrac{-2}{9}$	CLA, B.I.

Now the only task left is to check each of these solutions to make sure they work in the original equation. This is left for the reader to check.

SUMMARIZE STRATEGY FOR EQUATIONS WITH RADICALS

(a) First, isolate the term with the radical on one side of the equation. (If there is more than one radical term, at least isolate one of these)
(b) Kill the radical by squaring both sides of the equation, use Square Both Sides Theorem
(c) If there is another radical that has not been killed then isolate it, and again square both sides of the equation...
(d) Always check the solutions at the end to make sure you have not introduced extraneous solutions.

Excercises 6.2

(1) $\sqrt{x+1} = 4$
(2) $\sqrt{2x+5} = 5$
(3) $\sqrt{x-5} = \frac{x}{6}$
(4) $2\sqrt{2x+5} = 3\sqrt{3x-2}$
(5) $\sqrt{2x+5} = 1 + x$
(6) $\sqrt{2x-1} = 3 - 2\sqrt{3-2x}$
(7) $\sqrt{3x-2} + \sqrt{2x-1} = 2$
(8) $\sqrt{3x-2} + 2x + 1 = 3$
(9) $\sqrt{3x-2} + 2x = 5$
(10) $\sqrt{3x-2} + 2x + 5 = 5x - 2$
(11) $\sqrt{3x-2} + 2x = \sqrt{5}$

6.3. Absolute Value: Linear

"until one day... nothing happened"

Gameplan 6.3
(1) *AVT*
(2) *Strategy*
(3) *Practice*

Absolute Value Theorem [AVT]

$$\text{If } |x| = a \text{ then } x = a \text{ or } x = -a$$

Proof:

$a = \|x\|$	given
$\|x\| = \begin{cases} x & \text{if } x \text{ is non-negative,} \\ -x & \text{otherwise} \end{cases}$	Def of $\|x\|$
$a = x$ or $a = -x$	transitivity
$x = a$ or $x = -a$	B.I.

Examples: Practice Solving Equations

(1) Solve $|x + 3| = 4$

Solution:

$\|x + 3\| = 4$	Given
$x + 3 = 4$ or $x + 3 = -4$	AVT
$x = 1$ or $x = -7$	B.I.

Note, the reader is recommended to check that these solutions actually work. Without checking the most we know is that if the original equation has solutions, they have to be among these.

(2) Solve $|2x + 3| = 5$

Solution:

$\lvert 2x+3\rvert = 5$			Given
$2x+3=5$	or	$2x+3=-5$	AVT
$2x=2$	or	$2x=-8$	B.I.
$x=1$	or	$x=-4$	B.I.

Checking that each of these works is all there is left to do...

(3) Solve $\lvert 2x+3\rvert = -5$

Solution:

$\lvert 2x+3\rvert = -5$			Given
$2x+3=-5$	or	$2x+3=5$	AVT
$2x=-8$	or	$2x=2$	B.I.
$x=-4$	or	$x=1$	B.I.

Checking that each of these works is all there is left to do... but this time we check $x=-4$ and get $\lvert -5\rvert = 5$ which is false, so -4 is definitely not a solution. Then we check $x=1$ and get $\lvert 5\rvert = -5$ definitely not true, thus none of our candidates solves the equation. Therefore, there is no solution. Of course, a keen observer may have noted this from the beginning since the equation say that the absolute value of something is -5.. this is impossible...

Excercises 6.3

Solve!

(1) $\lvert 2x-5\rvert = 4$
(2) $\lvert 3x-5\rvert = 2$
(3) $\lvert 2x+8\rvert = 4+x$
(4) $\lvert 3x-5\rvert = -2$
(5) $\lvert 3x-5\rvert = -t$
(6) $\lvert 2x-5\rvert = 4x+2$
(7) $2(\lvert 2x+8\rvert) = 4+x$
(8) $-(\lvert 3x-5\rvert) = -2$
(9) $\lvert 2x-5\rvert = 4-x$
(10) $\left\lvert 3x^2-5x\right\rvert = 3x$
(11) $\left\lvert 3x^2-5x\right\rvert = 2x$
(12) $\lvert 2x+8\rvert = 4+x$
(13) $\lvert 3x-5\rvert = -2$
(14) $\lvert 2x-5\rvert = \lvert 4x+2\rvert$

6.4. Inequalities

"until one day... nothing happened"

Gameplan 6.4
(1) *Definitions and Axioms*
(2) *Famous Theorems*
(3) *Practice*

One Variable Inequalities, Definitions

(1) *Definition of* $<$

$$a < b \iff b - a \in \mathbb{R}^+$$

In words, we define "a is less then b" if and only if the difference $b - a$ is a positive real number.

(2) *Definition of* $>$

$$a > b \iff a - b \in \mathbb{R}^+$$

In words, we define "a is greater then b" if and only if the difference $a - b$ is a positive real number.

(3) *Addition Closure of Positive Reals*

$$a, b \in \mathbb{R}^+ \qquad \Longrightarrow \qquad a + b \in \mathbb{R}^+$$

In words, "if we add a positive plus a positive, we get a positive" this is an axiom, called the closure of positive reals $[+CloR^+]$ under addition.

(4) *Multiplication Closure of Positive Reals*

$$a, b \in \mathbb{R}^+ \qquad \Longrightarrow \qquad ab \in \mathbb{R}^+$$

In words, "if we multiply a positive times a positive, we get a positive" this is an axiom, called the closure of positive reals $[\times CloR^+]$ under multiplication.

(5) *Trichotomy Law* If $a \in \mathbb{R}$ then exactly one of the following is true...

$$0 < a \qquad \text{or} \qquad 0 = a \qquad \text{or} \qquad 0 > a$$

In words, this axiom says a real real number has to be greater than zero, less than zero, or equal to zero.

Famous Theorems

(1) *Cancellation Law of Addition for Inequalities (CLAI)*

$$a < b \qquad \Longrightarrow \qquad a + c < b + c$$

Proof:

$$\begin{aligned}
a < b &\quad \text{given} \\
b - a \in \mathbb{R}^+ &\quad \text{Def of } < \\
b - a + 0 \in \mathbb{R}^+ &\quad \text{Additive Identity} \\
b - a + c + -c \in \mathbb{R}^+ &\quad \text{Additive Inverses} \\
b + c + -c - a \in \mathbb{R}^+ &\quad \text{Commutativity Law of Addition} \\
(b + c) - (c + a) \in \mathbb{R}^+ &\quad \text{D.L.} \\
c + a < b + c \in \mathbb{R}^+ &\quad \text{Def of } <
\end{aligned}$$

(2) *Cancellation Law of Multiplication for Positive numbers (CLMP)*

$$a < b \quad \text{and} \quad 0 < c \implies \quad ac < bc$$

In words, this says we can multiply both sides of the inequality by any positive real number c. We will prove this one here. The others are left for the student to discover and prove.

Proof:

$$\begin{aligned}
a < b &\quad \text{given} \\
b - a \in \mathbb{R}^+ &\quad \text{Def of } < \\
c \in \mathbb{R}^+ &\quad \text{given} \\
(b - a)c \in \mathbb{R}^+ &\quad +CloR^+ \\
bc - ac \in \mathbb{R}^+ &\quad \text{DL} \\
ac < bc \in \mathbb{R}^+ &\quad \text{Def of } <
\end{aligned}$$

To the bank!!!

(3) *Cancellation Law of Multiplication for Negative numbers(CLMN)*

$$a < b \quad \text{and} \quad 0 > c \implies \quad ac > bc$$

In words, this says that if we multiply both sides of an inequality by a negative number, the inequality must be reversed. Try to prove this on your own. The proof is very neat, clean and elegant.

(4) *Transitivity Property of Inequalities (TPI)*

$$a < b \quad \text{and} \quad b < c \implies \quad a < c$$

(5) *Add Two Inequalities Theorem (ATIT)*

$$a < b \quad \text{and} \quad c < d \implies \quad a + c < b + d$$

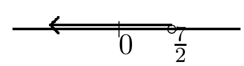
EXAMPLES

(1) Solve $x + 2 < 3x - 5$

Solution:

$$
\begin{array}{rcl}
x+2<3x-5 & & \text{given} \\
x+2+5<3x-5+5 & & \text{CLAI} \\
x+7<3x & & \text{B.I.} \\
x+-x+7<3x+-x & & \text{CLAI} \\
7<2x & & \text{B.I.} \\
\frac{1}{2}\cdot 7<\frac{1}{2}\cdot 2x & & \text{CLMP} \\
\frac{7}{2}<x & & \text{BI}
\end{array}
$$

Note: the solution can also be described in several other ways:

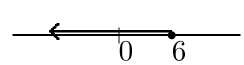
Set Notation	$\{x\in\mathbb{R}\mid\frac{7}{2}<x\}$
Interval Notation	$(-\infty,\frac{7}{2})$
Graph of Solution Set	

(2) Solve $-5(x-2)\geq -2(x+4)$

Solution:

$$
\begin{array}{rcl}
-5(x-2)\geq -2(x+4) & & \text{given} \\
-5x+10\geq -2x-8 & & \text{D.L.} \\
-5x+10+2x+-10\geq -2x-8+2x+-10 & & \text{CLAI} \\
-3x\geq -18 & & \text{BI} \\
\frac{1}{-3}\cdot -3x\leq \frac{1}{-3}\cdot -18 & & \text{CLMN**very important!!} \\
x\leq 6 & & \text{BI}
\end{array}
$$

Again, the solution can be represented in several other ways...

Set Notation	$\{x\in\mathbb{R}\mid x\leq 6\}$
Interval Notation	$(-\infty,6]$
Graph of Solution Set	

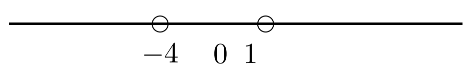
(3) Solve $x^2+3x-4<0$

Solution:

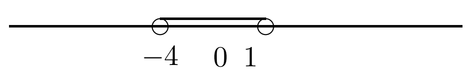
We approach this inequality in similar fashion. This time we split the Real Number line into several regions. There are 3 types of regions, those where $x^2 + 3x - 4 < 0$, those where $x^2 + 3x - 4 = 0$, and those where $x^2 + 3x - 4 > 0$. Again, the trick is to first plot the ones where $x^2 + 3x - 4 = 0$ because these separate the other regions. So we seek point were $x^2 + 3x - 4 = 0$, we shall call these *critical points.*

$$\begin{aligned} x^2 + 3x - 4 &= 0 && \text{Seek Critical Points} \\ (x + 4)(x - 1) &= 0 && \text{BI} \\ x + 4 = 0 \quad \text{or} \quad x - 1 &= 0 && \text{ZFT} \\ x = -4 \quad \text{or} \quad x &= 1 && \text{CLA} \end{aligned}$$

Thus, the critical points are $x = -4$ and $x = 1$. We now plot them, as they will serve to divide the real number line into *critical* regions.



Now, we've got our 3 regions, the region left of -4, the region between -4 and 1 and the region to the right of 1. We first test the region left of -4, by picking any point in that region, say $x = -5$. We check if $5^2 + 3 \cdot 5 - 4 < 0$. In deed $36 \not< 0$, so we conclude that none of the points in that region (left of -4) satisfy the inequality. We then move on to test the region strictly between -4 and 1, say $x = 0$. We check if $0^2 + 3 \cdot 0 - 4 < 0$ which is in fact true, so all of the points in that region satisfy the inequality. Then we move on to the third region, the one right of 1. We test $x = 3$, and check if $3^2 + 3 \cdot 3 - 4 < 0$ which is false, thus none of the point in that region satisfy the inequality. Finally, we check the critical points themselves, and realize that none of them work so we leave them as open circles. And summarize the solution set as $(-4, 1)$ or the graph of the solution:



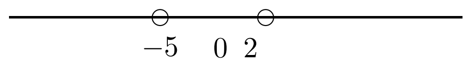
(4) Solve $x^2 + 5x \geq 2x + 10$

Solution:

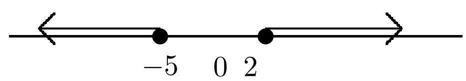
Again, we start by seeking the critical points. These are the points that divide the regions where $x^2 + 5x < 2x + 10$ and where $x^2 + 5x > 2x + 10$.

$$\begin{aligned} x^2 + 5x &= 2x + 10 && \text{Seek C.Pts} \\ x^2 + 5x - 2x - 10 &= 0 && \text{CLA} \\ x^2 + 3x - 10 &= 0 && \text{BI} \\ (x + 5)(x - 2) &= 0 && \text{BI} \\ x + 5 = 0 \quad \text{or} \quad x - 2 &= 0 && \text{ZFT} \\ x = -5 \quad \text{or} \quad x &= 2 && \text{CLA} \end{aligned}$$

We now plot the critical point to divide the real number line into critical regions...



Then we start testing regions. To test the left-most region we use the point $x = -6$ and check if $(-6)^2 + 5 \cdot -6 \geq 2 \cdot -6 + 10$. In deed, $36 - 30 \geq -16 + 10$ thus we conclude all of this region satisfies the inequality. We move on to check the middle region which does not work, and finally check the right-most region which does satisfy the inequality. At last we check the critical points and realize they too satisfy the inequality. We summarize with interval notation as $(-\infty, -5] \bigcup [2, \infty)$ or with a graph by:



EXCERCISES 6.4

Give solution as graph, interval, and using set notation

(1) $3x + 5 < -6x - 2$
(2) $3x + 5 > -6x - 2$
(3) $-3(x + 2) \leq -6(x - 2)$
(4) $-3x + \sqrt{10}) \geq -6x - 2$
(5) $2(x - 1) - 3(2x + 3) < -3x + \pi$
(6) $x^2 + 2x > 3x - 1$
(7) $x^2 + 2x \leq 3x - 1$
(8) $x(3x - 2) > 2x(x - 1)$
(9) $x(3x - 2) > 2x(3x - 8)$
(10) $x(-x + 2) < 2x(-3x - 8)$
(11) $x(-x + \pi) < 2x(-3x - 8)$

6.5. More Inequalities

CHAPTER 7

Graphing

7.1. The Slope of a Line

"until one day... nothing happened"

Gameplan 7.1
1. *What is a Graph*
2. *Slopes*
3. *Slope from Graph*
4. *Slope from 2 Points*

WHAT IS A GRAPH

When two quantities are related in some way, it is often desirable to know as much as possible the details of this relationship. Math makes it its business to develop the language and tools to this in a precise and effective manner. The two quantities could be mundane and trivial such as *earnings* and *hours* worked. Indeed, for most of us there is a precise relationship between how many hours we work and how much we earn. The mathematical language is to say that *earnings* is a *function* of *hours* worked. The concept of a function will be developed extensively later in our journey. For now we concentrate on describing the relationship between two quantities. In this case, the relationship between hours and earnings can also be described by a simple equation. Suppose, a particular soul is known to earn \$10/hour. Suppose we call the *number of hours worked* $'h'$, and the *amount earned* $'e'$. Then the relationship between e and h is given by the equation

$$e = 10h$$

Feel free to play around with this equation. Ponder for one second what happens if the amount of hours worked, h is 0? According to our clever equation, we substitute $h = 0$ and obtain the total earnings,

$$e = 10 \cdot 0 = 0$$

We could continue to probe the equation, if we work 1 hour, $h = 1$ then

$$e = 10 \cdot 1 = 10$$

If $h = 2$ then

$$e = 10 \cdot 2 = 20$$

In fact we could make a small chart describing the corresponding values for e as we pick various values for h.

h	0	1	2	3	4
e	0	10	20	30	40

Once we have this table of values we can represent them by a graph. Label the horizontal axis h for the amount of hours and the vertical axis e for earnings. We plot each of the above points, to obtain the following graph.

Upon close inspection, we note these points are all on some sort of straight line. Indeed they form a line, guided by these points we draw the line that connects them all. We call this line *the graph of* $e = 10h$, and it is a visual representation of the relationship between e and h.

It should be noted that the world is full of mundane quantities that are related in one way or another. For example, the number of hours spend doing homework is related (in theory) to your exam grades. Along the same lines, cab fairs are related to the number of miles travelled. The possibilities are truly endless. We will stop trying to keep track of them. Instead we simply call one quantity y and the other x. We leave their mundane interpretations to another day. Thus the above equation could have been written as

$$y = 10x$$

On the graphs it is customary to label the horizontal axis as x while the vertical axis is traditionally labelled y. The pairs of point are usually *ordered,* meaning the the *first* number is *always* the value of x while the second coordinate is always the value of y.
At the moment, we will concern ourselves exclusively with *linear equations,* meaning the highest degree in x and y is one. Incidentally, it is not a coincidence that *linear* equations all produce the graphs of *lines.* Let us take a close look at a second example. We will graph the equation

$$y = 3x - 2$$

To graph it we will study the relationship between x and y. Consider if $x = 0$ what is y? We simply plug in $x = 0$ to obtain..

$$
\begin{aligned}
y &= 3x - 2 && \text{given} \\
y &= 3 \cdot 0 - 2 && \text{sub } x = 0 \\
y &= -2 && \text{BI}
\end{aligned}
$$

This tells us that if $x = 0$ then $y = -2$ In other words the point $(0, -2)$ is on the line. We continue to find other point on the line that will give us a hint as to where the line should be graphed. We try to find what y is if $x = 1$

$$
\begin{aligned}
y &= 3x - 2 && \text{given} \\
y &= 3 \cdot 1 - 2 && \text{sub } x = 0 \\
y &= 1 && \text{BI}
\end{aligned}
$$

Thus the point $(1, 1)$ is also on the line. This time we try a negative number for x to see what y is. We will try $x = -1$ then ...

$$
\begin{aligned}
y &= 3x - 2 && \text{given} \\
y &= 3 \cdot^{-} 1 - 2 && \text{sub } x = 0 \\
y &=^{-} 3 +^{-} 2 && \text{BI} \\
y &=^{-} 5 && \text{BI}
\end{aligned}
$$

We continue this for a few more points, ultimately a very convenient way to keep track of these is to have a small table with x values and respective y values. Many of these you should be able to calculate *by inspection.*

x	-1	0	1	2	3
y	-5	-2	1	4	7

We are now ready to plot these points.

..and to connect them...

Slopes

Each line has a very important feature, namely it's steepness. The technical name for *the steepness of a line* is *the slope of the line.* We usually read lines from left to right. If the line is very steep *uphill* from left to right we say it has a very very high slope. If the line is flat, we say the line has slope 0. If the line is going downhill as we see it from left to right, we say it has a negative slope. If it goes downhill very fast we say it is has a very negative slope. Better than that, we can actually place a number on the 'steepness.' Consider the previous line we graphed, $y = 3x - 2$. Suppose we wanted to describe how steep the line is. One way to describe how steep it is, is to pick *any* two points on the line, say $(-1, 5)$ and a second point $(2, 4)$. We will focus on this part of the graph.

So we describe the steepness by saying go right 3, then up 9. It turns out that we can achieve the same steepness by going 1 right and 3 up. In other words, the ratio between the change the distance travelled in the x direction and the distance travelled in the y direction is at the heart of the steepness. For that reason we define the slope of a line as the quotient

$$steepness = \frac{change\ in\ y}{change\ in\ x}$$

There was once a famous mathematician who always misspelled steepness as *mteepness.* Because of this, the traditional variable to describe the steepness (slope) has been m. Then we can summarize our example. For the line $y = 3x - 2$ we picked two points $(-1, -5)$ and $(2, 4)$

and we from left to right we changed 3, while going up 9, thus the slope is

$$m = \frac{9}{3}$$

which can be reduced to

$$m = 3$$

Slope from Graph

Most of the time, from looking at the graph alone, one can determine the slope of a line. The only requirement is that one can read from the graph two different point on it. A few examples are in order.

(1) Find the slope for the following line.

The first task, is to find two points on the line. The challenge is to find point that go through coordinates we can read. The first obvious choice, by looking at the graph, is the point $(0, 4)$. This point is easy to read because the line goes exactly through that point on the graph. Another point we can read is the point $(-2, 7)$. There is nothing special about these points other than the fact that they are easy to read coordinates from. We now focus on these points, and calculate the change in y and the change in x. We will move left to right, from (-2,7) to (0,4). First, we move down 3 (note down 3 means -3), then right 2, for a total slope of...

$$m = \frac{\textit{change in } y}{\textit{change in } x} = \frac{-3}{2}$$

(2) We start over with a different line,

We pick two readable points, (-3,-8) and (3,2). As usual we move from left to right, thus first up 10 then right 6, for

$$m = \frac{\textit{change in } y}{\textit{change in } x} = \frac{10}{6} = \frac{5}{3}$$

SLOPE FROM 2 POINTS

The same sort of reasoning and computation should be doable without a graph. We now turn our attention to finding the slope directly from two points on a particular line. The principle is still the same

$$m = \frac{\textit{change in } y}{\textit{change in } x} = \frac{-3}{2}$$

This time we refine this definition just a bit. Suppose our first point it (x_1, y_1) and the second point is (x_2, y_2) then we can calculate the change in y is the second y minus the first y, in other words the change in y is $y_2 - y_1$. Similarly, the change in x is $x_2 - x_1$ Thus, we have an excellent definition of the slope base on definition of *slope*

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

(1) Find the slope from $(2, 3)$ to $(10, 7)$

Solution:

We first identify the first point $(x_1, y_1) = (2, 3)$ while the second point is $(x_2, y_2) = (10, 7)$ Then the slope is

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Def of } m \\ &= \frac{7-3}{10-2} && \text{Sub} \\ &= \frac{4}{8} && \text{BI} \\ &= \frac{1}{2} && \text{BI} \end{aligned}$$

(2) Find the slope from $(2, {}^{-}3)$ to $(10, 7)$

Solution:

We first identify the first point $(x_1, y_1) = (2, {}^{-}3)$ while the second point is $(x_2, y_2) = (10, 7)$ Then the slope is

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Def of } m \\ &= \frac{7-({}^{-}3)}{10-2} && \text{Sub} \\ &= \frac{10}{8} && \text{BI} \\ &= \frac{5}{4} && \text{BI} \end{aligned}$$

(3) Find the slope from $(2, {}^{-}3)$ to $(2, 7)$

Solution:

We first identify the first point $(x_1, y_1) = (2, {}^{-}3)$ while the second point is $(x_2, y_2) = (2, 7)$ Then the slope is

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Def of } m \\ &= \frac{7 - ({}^{-}3)}{2 - 2} && \text{Sub} \\ &= \frac{10}{0} && \text{BI} \\ &= not\, real && \text{BI} \end{aligned}$$

In fact, these points lie on a vertical line. The steepness is so great it is uncalculable by real numbers. It is customary to say this line has infinite slope or undefined slope.

(4) Find the slope from $(2, 7)$ to $(10, 7)$

Solution:

We first identify the first point $(x_1, y_1) = (2, 7)$ while the second point is $(x_2, y_2) = (10, 7)$ Then the slope is

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Def of } m \\ &= \frac{7 - 7}{10 - 2} && \text{Sub} \\ &= \frac{0}{8} && \text{BI} \\ &= 0 && \text{BI} \end{aligned}$$

Indeed, this shows that flat lines have no steepness. In other words, they have slope equal to 0.

We summarize this section with the important skills offered here. First, you should understand that a linear equation with two variable is an algebraic description of the relationship between the two variables. Second, you should be able to pick various values for x and find the corresponding values for y to construct a table of values, ultimately graphing the equation. You should understand what a slope is and how to produce it from a graph or from two points.

Classroom Exercises 7.1

(1) Graph

(a) $y = 3x + 1$
(b) $y = \frac{5}{2}x - 3$
(c) $y = -3x + 2$
(d) $y = 2$

(2) Find Slope for the line that goes through the two points

(a) $(2, 3)$ and $(5, 8)$
(b) $(2, -3)$ and $(5, 8)$
(c) $(2, 3)$ and $(5, -8)$
(d) $(2, 3)$ and $(5, 3)$

(3) Find Slope

EXCERCISES 7.1

(1) Graph

(a) $y = -3x + 1$
(b) $y = \frac{-5}{2}x - 3$
(c) $y = 3x + 3$
(d) $x = 2$

(2) Find Slope for the line that goes through the two points

(a) $(2, 3)$ and $(3, 8)$
(b) $(2, -3)$ and $(-2, 8)$
(c) $(-2, -3)$ and $(5, -8)$
(d) $(-2, 7)$ and $(-2, 3)$

(3) Find Slope

7.2. The Equation/Graph of a Line

"until one day... nothing happened"

Gameplan 7.2
(1) *Slope Intercept Form [SI]*
(2) *Point Slope Form [PS]*
(3) *Put it All Together*

Slope Intercept Form [SI]

A linear equation in two variable may take on various forms. All of the equations below are indeed linear equations.

(1) $3x + 2y = 87$
(2) $x = 2 + 3y$
(3) $3(x - 2) = 6(2 + y)$
(4) $y = 2x + 1$

Yet we will make an important distinction. The *Slope-Intercept [SI]* form of a linear equation is the form,

$$y = mx + b$$

where m is some number and, b is some other number. For example, the equation

$$y = 3x + 5$$

is in fact written in *slope-intercept* form. Where the $m = 3$ and $b = 5$.
The slope intercept form of an equation make is very easy to tell the slope of the equation. Think about it for a second. If we increase x by one then y will increase by whatever coeeficent x has. In the above example, if we construct a table of values we see the following...

x	0	1	2	3	4
$y = 3x + 5$	5	8	11	14	17

Every time we increase x by 1, y increases by 3. Then the slope of the line is given by

$$m = \frac{\textit{change in } y}{\textit{change in } x} = \frac{3}{1} = 3$$

And in general, every time x changes by one, $y = mx + b$ will change by m. Thus, if a slope exists it will be

$$m = \frac{\textit{change in } y}{\textit{change in } x} = \frac{m}{1} = m$$

In addition, b also tells us something very special. Consider the value for y in $y = mx + b$ if we set $x = 0$.

$$y = m\ t0 + b = b$$

This tells us that the point $(0, b)$ is always part of the line. In other words, the line crosses the y axis exactly at the point $y = b$. This warrants the name for b, the "y-intercept". We can exploit this idea to say something about our previous equation $yu = 3x + 5$ Namely, that it crosses the y-intercept at $y = 5$ and since we've already calculated the slope to be 3, we can readily graph the line.

We first plot the y-intercept,

From that point, we draw slope 3, 3 up and 1 right, to obtain the line,

Slope Intercept Form [SI]

$$y = mx + b$$

m, the coefficient of x is the slope of the line .

b, called the y-intercept, tells where the line crosses the y-axis

We can repeat this very efficiently. Consider a second example, to graph

$$y = \frac{-2}{3}x + 3$$

We first mark the $y - intercept$ at $y = 3$ and from there draw the slope, $\frac{-2}{3}$, down 2 and right 3.

POINT SLOPE FORM [PS]

We've seen how easy [SI] makes it to graph equations. We now turn our attention to *Point Slope [PS]* to explore some of its nice features. Consider the difference between the graphs for equations $y = x$ and $y = x - 2$. The difference is clear, they both have the same slope, $m = 1$, yet, the second equation, $y = x - 2$ gets shifted right 2.

$y = x \qquad y = x - 2$

The above, is a prelude to a famous 'shifting' principle which we will fully exploit in due time. This shifting principle works just as swiftly vertically. Consider, replacing x with $x - 2$ shifted the graph right 2. One would then expect if we replace y with $y - 3$ the graph would shift (vertically) up 3. This shifting principle is good everywhere. The respective graphs are

$y = x \qquad y - 3 = x$

The shifting principle works in every possible direction. If we replace x with $x + 3$ the shift would go left 3, rather than right. The principle works so well we can easily generalize it as the key feature for the *Point Slope [PS]* form of a linear equation.

Point Slope Form [PS]

$$y - k = m(x - h)$$

$y - k$ describes the vertical shift

$x - h$ depicts horizontal shift

With vertical shift k units, horizontal shift h units the line must go through the point (h, k)

Suppose we wanted to graph the equation

$$y - 1 = \frac{-2}{5}(x - 3)$$

Then is becomes clear that line is shifted up 1 and right 3, thus it must go through the point $(3, 1)$. We mark the spot, then from there, draw the steepness, $m = \frac{-2}{5}$.

mark the point $(3, 1)$ draw the slope $\frac{-2}{5}$

Suppose this time we wanted to graph the equation

$$y - 1 = 3(x + 5)$$

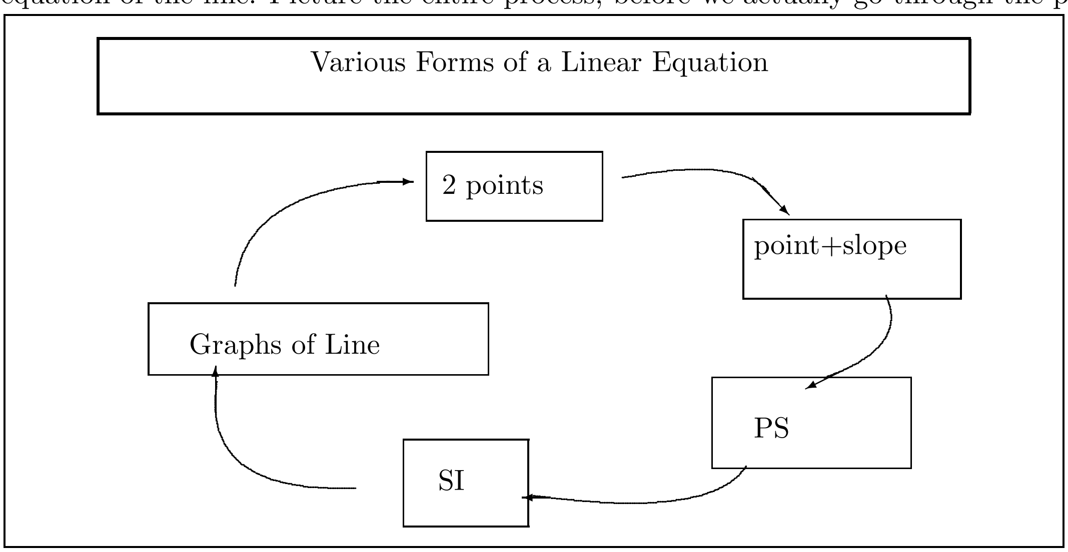
As it stands, the equation looks *almost* like the Point Slope form. The only difference is that the right side does not take the form $m(x - h)$ it takes the form $3(x + 5)$. However, we can change by using the fact that a negative times a negative is a positive. Thus, we can re-write the above equation as

$$y - 1 = 3(x - {}^{-}5)$$

Then is becomes clear that line is shifted up 1 and left 5, thus it must go through the point $(-5, 1)$. We mark the spot, then from there, draw the steepness, $m = 3$.

Put it All Together

We should be able to go through all phases and forms of a linear equation. Starting from two given points on a line, we can determine the slope, using the definition of slope. We can then, take any one of these two points and the computed slope to obtain the PS form. From the PS form we can solve for y to obtain the SI form. From the SI form we can easily graph the equation of the line. Picture the entire process, before we actually go through the process.



We end this section byu carrying out the above cycle for the line that goes through the points $(3, 5)$ and $(-2, -5)$.

2 Points → Point+Slope

Using the definition we first compute the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Def of } m \\ &= \frac{-5-5}{-2-3} && \text{Sub} \\ &= \frac{-10}{-6} && \text{BI} \\ &= \frac{5}{3} && \text{BI} \end{aligned}$$

Once we have the slope, we can pick any of the two point. We pick $(3, 5)$

POINT+SLOPE → PS

Since we know the point $(3, 5)$ is on the line and $m = 5/3$ we can write the PS form, generally PS takes the form

$$y - k = m(x - h)$$

In this case, $(h, k) = (3, 5)$ Therefore, the PS form for this equation is

$$y - 5 = \frac{5}{3}(x - 3)$$

PS → SI

To change from PS to SI, we need only solve for y so as to write in the $y = mx + b$ form.

$$\begin{aligned} y - 5 &= \frac{5}{3}(x - 3) && \text{Given PS} \\ y - 5 &= \frac{5}{3}x - \frac{5}{3} \cdot 3 && \text{DL} \\ y - 5 &= \frac{5}{3}x - 5 && \text{DL} \\ y &= \frac{5}{3}x + 0 && \text{CLA} \end{aligned}$$

And now we have SI form, note the y-intercept is 0

$$y = \frac{5}{3}x + 0$$

SI → GRAPH OF LINE

Recall to graph an equation in SI form, we first mark the y-intercept then draw the slope from there...

mark the y-intercept $y = 0$ draw the slope $\frac{5}{3}$

EXCERCISES 7.2

(1) Given points $(2, 6)$ and $(3, 10)$ on a line:

(a) Find the slope, m
(b) Write PS form of Equation
(c) Write SI form of Equation
(d) Graph the Equation

(2) Given points $(2, 6)$ and $(6, -2)$ on a line:

(a) Find the slope, m
(b) Write PS form of Equation
(c) Write SI form of Equation
(d) Graph the Equation

(3) Find the slope for each of the lines.

(a) $y = -3x + 2$
(b) $y = -\frac{2}{3}x + 2$
(c) $y = 5(x + 2)$
(d) $y = 2$
(e) $4y + 7x = 2$
(f) $x = 2$

(4) Graphs each of the Equations

(a) $y = -3x + 2$
(b) $y = -\frac{2}{3}x + 2$
(c) $y = 5(x + 2)$
(d) $y = 2$
(e) $x = 2$

Classroom Exercises 7.2

(1) Given points $(2, -6)$ and $(3, 10)$ on a line:

(a) Find the slope, m
(b) Write PS form of Equation
(c) Write SI form of Equation
(d) Graph the Equation

(2) Find the slope for each of the lines.

(a) $y = -3x + 2$
(b) $y = -\frac{2}{3}x + 2$
(c) $4y - 3x = 2$
(d) $y = 2$
(e) $x = 2$

(3) Graphs each of the Equations

(a) $y = -x + 2$
(b) $y = -\frac{1}{3}x + 2$
(c) $y = 2(x + 2)$
(d) $y = -2$
(e) $x = 5$

7.3. The Graph of a Parabola & Shifting Principle

"until one day... nothing happened"

Gameplan 7.3

(1) *Parabola Basics*
(2) *Shifting Principle*
(3) *Complete the Square*

Parabola Basics

Equations which are degree one in y and degree two in x all take the same general shape upon graphing. The shape is generally a 'u' shape and is commonly referred to as a *parabola.* Consider graphing the most basic parabola,

$$y = x^2$$

One favorite approach to graphing this equation is to begin plotting points.

x	-3	-2	-1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9

plot points ... then connect them

We could then toy with the idea of changing the quadratic coefficient on x^2. Recall for linear equations, the coefficient of x determined the slope or steepness of the graph. It should not come to a surprise that a similar phenomena is observed here.

Example

Consider graphing

$$y = 2x^2$$

We will plot points to graph this.

x	-3	-2	-1	0	1	2	3
$y = 2x^2$	18	8	2	0	2	8	18

Notice the change in steepness. One way to measure the steepness of this curve is to check how fast it climbs going from $x = 0$ to $x = 1$. Note the original graph $y = x^2$ goes from the point $(0, 0)$ to the point $(1, 1)$. In other words, one the first right causes 2 units up. Meanwhile for our new parabola, $y = 2x^2$, the first unit right causes 2 units up, going from $(0, 0)$ to $(1, 2)$. Of course we do not intend to suggest that every unit to right cause 2 units up, rather that the *first* unit right causes 2 units up.

Example

As a third example, consider graphing

$$y = 3x^2$$

As usual, we construct a table of values, plot them and connect them.

x	-2	-1	0	1	2
$y = 3x^2$	12	3	0	3	12

Notice the change in steepness. For our new parabola, $y = 3x^2$, the first unit right causes 3 units up, going from $(0, 0)$ to $(1, 3)$.

Shifting Principle

For linear equations, we discussed the ramification on the graph when x is replaced by $x-h$. Namely, the graph gets shifted horizontally h units, while replacing y with $y-k$ results in a vertical shift k units. The same principle holds here for parabolas. In fact, this shifting principle holds everywhere, for every function, on every graph, on any day, by land or by sea. For now, we will exploit this shifting principle to discuss a general form of a parabola.

General Vertical Parabola

$$y - k = m(x-h)^2$$

$y-k$ describes the vertical shift

$x-h$ depicts horizontal shift

With vertical shift k units, horizontal shift h units the parabola must go through its vertex point, (h,k).

Example

Consider graphing

$$y + 3 = 3(x+2)^2$$

One possible plan of attack is to plot points. Here we opt to take advantage of the shifting principle, and the general form described above. On the equation the left side, $y+3$, describes a vertical shift down 3, while $x+2$ describes a shift left 2. The coefficient 3 still describes the steepness, namely the first unit right causes 3 units up. Thus a good plan of attack is to first find the vertex (the 'center' of the parabola), in this case down 3 and left 2 puts the vertex at $(-2,-3)$.

From there, we can complete the graph by reasoning, right one up 3, and left one, up 3, then complete the graph. It should be noted that sometimes the coefficient m may be negative. A negative m indicates a parabola that opens downwards, rather than upwards. In addition, equations sometimes have to re-written in standard form so as to immediately identify the shifts, and the steepness. The following is an excellent example. It demands re-writing the equation first, then exploring the idea of a negative m.

Example

Graph

$$y + 3x^2 = -6x - 2$$

Solution:

First note the equation is degree 2 in x and degree one in y thus a bonafied parabola. Note the standard form of a parabola equation has a perfect square (PP2) on the right side. We will need move the x to right side and complete the square.

$$\begin{aligned} y + 3x^2 &= -6x - 2 && \text{g} \\ y + 2 &= -3x^2 - 6x && \text{CLA} \\ y + 2 &= -3(x^2 + 2x) && \text{DL} \\ y + 2 +^{-} 3 \cdot 1 &= -3(x^2 + 2x + 1) && \text{CLA} \\ y +^{-} 1 &= -3(x + 1)^2 && \text{PP2} \end{aligned}$$

And now we are ready. The vertex is $(-1, 1)$, the -3 points to a downward parabola with 'steepness' 3.

Excercises 7.3

(1) Graph $y = 2x^2$
(2) Graph $y = 4x^2$
(3) Graph $y = -2.5x^2$

(4) Graph $y - 3 = 2(x - 2)^2$
(5) Graph $y - 2 = -(x + 3)^2$
(6) Graph $y + 5 = (x - 3)^2$
(7) Graph $y = -2x^2 + 8x$
(8) Graph $y - 2 = x^2 + 6x$
(9) Graph $y = x^2 - 2x$

7.4. The Graph of a Circle & Shifting Principle

"until one day... nothing happened"

Gameplan 7.4
(1) *Circle Basics*
(2) *Shifting Principle*
(3) *Complete the Square*

CIRCLE BASICS

The equation of the basic circle is

$$x^2 + y^2 = 1$$

The defining features are: degree 2 in x, degree 2 in y both coefficients on x^2 and y^2 are equal and positive, and 1 on the right side indicates the radius. The fact that the graph of this famous equation is a circle may not be completely clear at first. We can study the question by simply plotting points. Before we plot points, we take a moment to solve for y so as to streamline the process.

$$\begin{aligned} x^2 + y^2 &= 1 && \text{given} \\ y^2 &= 1 - x^2 && \text{CLA} \\ y &= \pm\sqrt{1 - x^2} && \text{SRP} \end{aligned}$$

This means that for every x value we two possible y's to consider, a positive and a negative version. For example, if we set $x = 0$, then we obtain $y = \pm\sqrt{1 - 0^2} = \pm 1$. This means that both point $(0, 1)$ and $(0, -1)$ are on the graph. We try other values for x to complete a reasonable table. From there we plot the points and connect them.

x	-1	-.75	-.5	-.25	0	.25	.5	.75	1
$y = \pm\sqrt{1 - x^2}$	± 0	$\pm .66$	$\pm .86$	$\pm .96$	± 1	$\pm .96$	$\pm .86$	$\pm .66$	± 0

MORE CIRCLE BASICS

We now consider the general form of a circle. We can experiment with the above equation to see what happens we replace 1 with a larger number like 4 or 25. After a few tables with points and a few plotted graphs we may be able to discover the pattern. Namely, the right hand side gives the radius of the circle, assuming it is positive. In other words, the equation of a circle with radius R is given by

$$x^2 + y^2 = R^2$$

Yet, we have the necessary tools to write a good proof. We first construct the formal statement and then we will expose the proof.
If the point (x, y) lines on a circle with positive radius R, then

$$x^2 + y^2 = R^2$$

For a proof consider any point, (x, y), on a circle with radius R. We can drop a perpendicular to draw a triangle. We can then use pythagoras to conclude...

$$x^2 + y^2 = R^2$$

EXAMPLE

Graph the equation

$$x^2 + y^2 = 9$$

Solution:

Note this the standard for of the equation of a circle with the square of the radius being 9. Said another way the radius is the square root of 9 which is 3. Once we know the radius, we simply draw the graph. It may be helpful to plot the center of the circle, in this case $(0, 0)$. In addition, it may be helpful to mark at least four obvious points on the circle, from the center, we mark the point 3 units up, down, right, and left. Use these points to finish the graph.

$$x^2 + y^2 = 9$$

Shifting Principle on Circles

Here is yet one more instance of brilliance by the *shifting principle.* It works just as swiftly and consistently as it did for lines and parabolas. Specifically, replacing x by the quantity $(x - h)$ results in a horizontal shift h units, while replacing y with $y - k$ results in a vertical shift k units. When shifting a circle it may be helpful to shift the center, then draw the circle around the center.
We summarize the shifting principle on circle below.

General Circle Equation

$$(y - k)^2 + (x - h)^2 = R^2$$

$y - k$ describes the vertical shift

$x - h$ depicts horizontal shift

The center shifted to (h, k), to determine the radius, take the square root of the right side, $\sqrt{R^2} = R$

Example

Graph the equation

$$(y-3)^2+(x+1)^2=49$$

Solution:

We should talk about it first. The $y-3$ piece describes a shift up 3, while $x+1$ describes a shift left 1. The radius is the square root of the right side, which is 7, since $\sqrt{49}=7$. So a good way to start the graph is to start with the center. Mark the center spot at $(-1,3)$. From the center, mark the spots: up 7, left 7, down 7, and right 7, then finish the graph. The final sketch should resemble this:

$$(y-3)^2+(x+1)^2=49$$

EXAMPLE

Graph the equation:

$$x^2-6x+y^2+4y=0$$

Solution:

The immediate obstacle is that the equation is not in standard form. To write it in standard form we can *complete the square* on the x terms as well as on the y terms. You may recall, completing the square is synonymous with completing a PP2 polynomial. We proceed,

$$\begin{aligned}
x^2 - 6x + y^2 + 4y &= 0 && \text{given}\\
x^2 + 2 \cdot {}^{-}3x + y^2 + 2 \cdot 2y &= 0 && \text{BI (beggining of PP2)}\\
x^2 + 2 \cdot {}^{-}3x + ({}^{-}3)^2 + y^2 + 2 \cdot 2y + 2^2 &= ({}^{-}3)^2 + 2^2 && \text{CLA}\\
(x + {}^{-}3)^2 + (y+2)^2 &= ({}^{-}3)^2 + 2^2 && \text{PP2}\\
(x + {}^{-}3)^2 + (y+2)^2 &= 13 && \text{PP2}
\end{aligned}$$

Having completed the square we can see what the shifts are as well as what the radius is. The shifts are right 3 and down 2, placing the center at $(3, -2)$. The radius is the square root of the right side, $\sqrt{13} \approx 3.6$. We now have all the information necessary to sketch the graph of the circle.

$$(x + {}^{-}3)^2 + (y+2)^2 = 13$$

EXCERCISES 7.4

(1) Graph the following circles.

(a) $x^2 + y^2 = 4$
(b) $x^2 + y^2 = 36$
(c) $x^2 + y^2 = 64$
(d) $x^2 + y^2 = 20$

(2) Graph the following circles.

(a) $(x+3)^2 + y^2 = 4$
(b) $(x+3)^2 + (y-1)^2 = 36$
(c) $(x-1)^2 + (y-3)^2 = 64$
(d) $(x+3)^2 + (y+3)^2 = 20$

(3) Graph the following circles.

(a) $y^2 + 10y + x^2 + 6x = 3$

(b) $y^2 + 10y + x^2 + 5x = 3$

7.5. Graph Hyperbolas and Ellipses

"until one day... nothing happened"

Gameplan 7.5
(1) *Ellipse Basics*
(2) *Hyperbola Basics*
(3) *Shifting Principle*
(4) *Complete the Square*

Ellipse Basics

The graph of an ellipse takes an oval shape. It looks very much like the graph a circle, except it its no necessarily perfectly round. Since ellipses are so much like circles in their graphs, it should not come as a surprise that their equations are much like the ones for circles. Indeed, the equation of an ellipse is characterized by the sum of one degree two in x and one degree two term in y, just like the equation of a circle. Yet, the difference is that the coefficients need not be equal. In other words, the equations $x^2 + y^2 = 1$ and $\frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$ both show the sum of degree two terms in x and y. However, the second equation shows a coefficient of $\frac{1}{4}$ on the degree two $x-$term, while the y-term has coefficient $\frac{1}{9}$. This makes the second equation an ellipse, while the first is a circle. It is customary to show any coefficients as a fraction with the quadratic term as a numerator. The general form for the equation of an ellipse is given by:

General Ellipse Equation

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

$y - k$ describes the vertical shift

$x - h$ depicts horizontal shift

The center shifted to (h, k), the denominator under y^2 determines the vertical stretch factor of a, while under the x^2 we find the horizontal stretch factor b right side

Examples: Graphing Ellipses

(1) Graph $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Solution:

The reader is invited to plot points as an alternative way to graph the equation. Here will will opt for a small generalization of the shifting principle. Note the shifting principle is the observation that replacing x by $x-2$ results in a shift of 2 rightward. Similarly, we observe a *stretching principle.* Replacing x with $\frac{x}{2}$ results in a stretching of the graph by a factor of two. Similarly, replacing y with $\frac{y}{3}$ with result in a veritcal stretch by a factor of 3. Thus, if an equation begins as $x^2+y^2=1$ then the above replacements are made on this equation, the resulting equation is

$$\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$$

which then becomes

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

which is exactly the equation we want to graph. We summarize, start with a circle radius 1 and equation $x^2+y^2=1$.

We then modify the equation, resulting in a horizontal stretch by 2 and a vertical stretch by 3, and the new equation, $\frac{x^2}{4}+\frac{y^2}{9}=1$

(2) Graph $\frac{(x-3)^2}{4} + \frac{(y+1)^2}{9} = 1$

Solution:

A close inspection reveals striking similarities between this and the previous equation. Indeed, the only difference is instead of x^2 we have $(x-3)^2$ which results in a shift right 3 units while the y^2 replaced by $(y+1)^2$ results in a vertical shift, down 1. The denominators are still 4 and 9, thus the vertical and horizontal stretch will remain the same. With all this analysis done, we wrap things up by simply finding the new center of the ellipse. We find the center shifted right 3, down 1. Once there, we will go left and right 2 and up and down 3 marking four points on the ellipse. The result...

HYPERBOLAS

We continue to graph equations of degree two in each x and y. This case we consider the case where corresponding degree-two coefficients have opposite signs. These equations yield curves called hyperbolas. The equations resemble those of ellipses with the only distinction being the difference of the sign on either the quadratic coefficients. The general (horizontal) hyperbola equation takes the form:

General Hyperbola Equation

$$\frac{(x-h)^2}{b^2} - \frac{(y-k)^2}{a^2} = 1$$

$y - k$ describes the vertical shift

$x - h$ depicts horizontal shift

The center shifted to (h, k), the denominator under y^2 determines the vertical stretch factor of a, while under the x^2 we find the horizontal stretch factor b right side

Again, we take this opportunity to put a plug in for the *plotting points* method of graphing. If you have never seen the graph of an equation of this form before, you are strongly urged to plot points at least a few times. We will take this advise and plot points for the most basic hyperbola. The most basic hyperbola is given by the equation

$$x^2 - y^2 = 1$$

After plotting many points we obtain the basic graph.

Notice we can use these "corners" as guide to complete the curve. By considering large values for x and y we may find some way to describe the behavior of the graph. A careful consideration of these ideas is not within the scope of this course. Yet the finding are very accessible to us. It turns out that we can draw diagonals through these corners to serve as guides to complete the graph as the values of x increase or decrease. These lines are called *oblique asymptotes.* In short, Every graph of a hyperbola can start with these 4 corners, then the diagonals, and finally, we use these diagonals to guide the sketching of the hyperbola. Again , the shifting and stretching principal apply as before. A couple examples are in order.

Examples: Graphing a Hyperbola

(1) Graph $\frac{x^2}{4} - \frac{y^2}{9} = 1$

Solution:

We first note there are no shifts on this hyperbola. There is a horizontal stretch factor of 2 and a vertical stretch factor of 3. This is already enough to plot the center and the four corners, and the diagonal guides.

Once we've done this much, the rest is easy. Just sketch the hyperbola using the inside 'box' and the diagonals as guides. The final graph is...

(2) Graph $\frac{y^2}{9} - \frac{x^2}{1} = 1$

Solution:

One major difference here is the x^2 coefficient is the negative one, not the y^2. The hyperbola will open vertically, rather than horizontally. The rest of the process will be the same. We begin by finding the center at $(0, 0)$. We stretch the 'box' left/right 1, and up/down 3, then plot the corners and the diagonals. Thus far we have

We then sketch the hyperbola (vertically) to obtain:

(3) Graph $\frac{(x-3)^2}{4} - \frac{(y+1)^2}{9} = 1$

Solution:

Note how much this equation resembles the previous ellipse example. The only difference is the minus sign between the degree two terms. Also note the process to graph if will be almost identical until the very end when instead of drawing the ellipse through the for 'corner' points we will draw diagonals and draw the hyperbola guided by the diagonals.

Note the center is $(3, -1)$ from there, we will move up/down 3 and left/right to create our guiding box. We then draw the our diagonal guides to obtain...

When then observe the coefficient on y^2 is negative, therefore it is a horizontal parabola opening left and rightward, we finish with the sketch....

(4) Graph $9x^2 - 4y^2 - 54x - 8y = -41$

Solution:

This equation looks very different than the previous ones. Yet some key features remain the same. It is still degree two in x and degree two in y with different signed coefficients. This makes it a certified hyperbola. The other hyperbolas were easy to graph because the equations were given in standard form. If this equation was written in standard form we would be ready to graph it, but it is not. This will not stop us. We will simply re-write it in standard form. In a nutshell, all we need to do to write it in standard form is to complete the square. Talk is cheap, let us begin.

$$\begin{aligned}
9x^2 - 4y^2 - 54x - 8y &= -41 && \text{given}\\
9x^2 +^- 4y^2 +^- 54x +^- 8y &=^- 41 && \text{Ded } a-b\\
9x^2 +^- 54x +^- 4y^2 +^- 8y &=^- 41 && \text{CoLA}\\
9(x^2 +^- 6x) +^- 4(y^2 + 2y) &=^- 41 && \text{DL}\\
9(x^2 + 2 \cdot^- 3x) +^- 4(y^2 + 2 \cdot 1y) &=^- 41 && \text{BI}\\
9\left(x^2 + 2 \cdot^- 3x + (^-3)^2\right) +^- 4(y^2 + 2 \cdot 1y + 1^2) &=^- 41 + 9(^-3)^2 +^- 4(1^2)\cdot && \text{CLA}\\
9(x +^- 3)^2 +^- 4(y+1)^2 &=^- 41 + 81 +^- 4 && \text{BI}\\
9(x +^- 3)^2 +^- 4(y+1)^2 &= 36 && \text{BI}\\
\frac{9(x +^- 3)^2}{36} + \frac{^-4(y+1)^2}{36} &= \frac{36}{36} && \text{CLM, DL}\\
\frac{(x +^- 3)^2}{4} - \frac{(y+1)^2}{9} &= 1 && \text{CLM, DL}
\end{aligned}$$

Now that is much better. The above is called 'standard form' for the equation of a hyperbola. Moreover, we recognize it as the hyperbola graphed a couple examples ago. Refer back to it to see the final graph.

Excercises 7.5

(1) Graph $\frac{x^2}{25} + \frac{y^2}{4} = 1$
(2) Graph $\frac{x^2}{25} - \frac{y^2}{4} = 1$
(3) Graph $\frac{x^2}{25} + \frac{y^2}{4} = 1$
(4) Graph $\frac{(x+3)^2}{4} + \frac{(y+1)^2}{64} = 1$
(5) Graph $\frac{(x+3)^2}{4} - \frac{(y+1)^2}{64} = 1$
(6) Graph $4x^2 + 25y^2 = 100$
(7) Graph $\frac{(x-5)^2}{9} + \frac{(y-1)^2}{64} = 1$
(8) Graph $\frac{(x-5)^2}{9} - \frac{(y-1)^2}{64} = 1$
(9) Graph $\frac{x^2}{3} + \frac{y^2}{5} = 1$
(10) Graph $\frac{x^2}{\frac{1}{4}} + \frac{y^2}{\frac{1}{25}} = 1$
(11) Graph $4x^2 + 25y^2 = 1$

7.6. The Graph Misc

"until one day... nothing happened"

Gameplan 7.6
(1) *Review Graphing Principles*
(2) *Various Graphs*

Review of Graphing Principles

Principle	Equation/Graph	Resulting Equation/Graph
Shifting Principle This has no exceptions, it is as consistent as sunrise, **replacing** x **with** $x-a$ shifts the graph a units. Note the shifting principles works identically with respect to the y axis.	$x^2+y^2=1$	$(x-3)^2+y^2=1$
Stretching Principle This has no exceptions, it is as consistent as consistent can be. **replacing** y **with** $\frac{1}{3}y$ stretches the graph vertically by a factor of 3. This means each y value is multiplied by 3. Note this principle applies to horizontal stretching as well	$x^2-y^2=1$	$x^2-\left(\frac{y}{3}\right)^2=1$
Reflecting Principle This has no exceptions, it is as consistent as mustard. **replacing** x **with** $-x$ flips the graph over the y-axis. This means each x value is multiplied by $-x$. Note this principle applies vertically as well.	$y=(x-3)^2$	$y=(-x-3)^2$

Miscellaneous Graphs

During the last couple of sections we have studied very special and famous graphs like those of circles, ellipses, hyperbolas, lines, and parabolas. These are collectively known as conics (think about why the name 'conics'). Having done all the famous ones we now turn to the less famous. The idea here is to sample a wide range of graphs while still practicing the graphing principles. See the examples below.

(1) Graph $y = |x|$

Solution:

As usual, our first meeting with any equation should be immediately followed by a plotting-points session. We find:

x	-3	-2	-1	0	1	2	3
$y = \|x\|$	3	2	1	0	1	2	3

We now plot these to present the final graph, the famous V-shape.

(2) Graph $y + 2 = |x + 3|$

Solution:

This equation is exactly like the previous one, except for replacing y with $y + 2$ and replacing x with $x + 3$. But we know exactly what these replacements do the graph. The graph will be shifted down 2 and right 3. No need to plot points here, just shift the previous graph. The new graph is given as

(3) Graph $y = \sqrt{x}$

Solution:

As usual, our first meeting with any equation should be immediately followed by a plotting-points session. We find:

x	-3	-2	-1	0	1	4	5
$y = \sqrt{x}$	not real	not real	not real	0	1	2	2.23

We now plot these to present the final graph, the famous V-shape.

(4) Graph $y = \sqrt{x+3}$

Solution:

We look at the previous graph, then shift left 3..

(5) Graph $-\frac{1}{2}y = \sqrt{x+3}$

Solution:

We look at the previous graph, nd previous equation. The y was replaced by $-\frac{1}{2}y$. The negative flips the graph over the x axis. The $\frac{1}{2}$ stretches the graph vertically with a factor of 2, thus every y value is multiplied by two.

Excercises 7.6

(1) Graph $y = |2x|$
(2) Graph $y = |2(x+3)|$
(3) Graph $-y = |2x|$
(4) Graph $y - 2 = \sqrt{x}$
(5) Graph $y = \sqrt{-x}$
(6) Graph $y = \sqrt{-(x-2)}$
(7) Graph $y = \sqrt{-x+2)}$
(8) Graph $y = 2^x$
(9) Graph $y = 2^{x+3}$
(10) Graph $y = 2^{x-1}$
(11) Graph $-y = 2^x$
(12) Graph $y = 2^{-x}$
(13) Graph $y = \frac{1}{x}$
(14) Graph $y = \frac{1}{x-3}$
(15) Graph $y - 3 = \frac{1}{x}$
(16) Graph $y = \frac{1}{x} + 3$

CHAPTER 8

Solving Systems

8.1. Introduction

"until one day... nothing happened"

Gameplan 8.1
(1) *What is a system*
(2) *What is a solution*
(3) *Who has solution, how many?*
(4) *Check solutions*

SYSTEMS

We will define *a system of equations* as a bunch (one or more) of equations. A linear system is one where all equations are linear. A non-linear system is one where one or more of the equations is non-linear.
We define a solution as a set of values, one for each of the variables, such that these values satisfy all equations in the system.
Not all systems have a solution. For linear systems, there are 0, 1 or infinite many solutions. For a linear system, there will never be exactly 2 solutions for example. If a system has 0 solutions it is called and *inconsistent system.* If it has exactly one solution it is called a *consistent system.* If it has infinite many solutions it is called *dependent system.*

CHECKING A SOLUTIONS

Here the idea is to practice what we mean when we say we have a solution.

(1) Check to see if $x = 1, \qquad y = -3$ is a solution to the following system.

$$x + y = -2$$
$$x - y = -4$$

Solution:

We check the first equation: $1 + -3 = -2$ is true.
We check the second equation: $1 + -(-3) = -4$ is not true!
Thus, $x = 1, y = -3$ is not a solution to this system.

(2) Check to see if $x = -1, y = 3$ is a solution to the following system.

$$x^2 + 4 = 2y - 1$$
$$x + y = -2$$
$$2x^3 - 3y = -7$$

Solution:

We check the first equation: $(-1)^2 + 4 = 2(3) - 1$ is true.
We check the second equation: $-1 + 3 = -2$ is true.
We check the last equation: $2(-1)^2 - 3(3) = -7$ is also true.
Thus, $x = -1, y = 3$ is a solution to this system. Note the solution is often represented with ordered pairs such as $(-1, 3)$

EXCERCISES 8.1

(1) Check if $(1, -2)$ solves.

$$x + y = -1$$
$$x - y = -4$$

(2) Check if $(1, -2)$ solves.

$$2x + y = 0$$
$$5x - y = 7$$

(3) Check if $(1, -2)$ solves.

$$2x^2 - 5x + y^2 = 1$$
$$5x^3 - y^2 = y + 3$$

(4) Check if $(1, 1)$ solves.

$$2x^2 - 5x + y^2 = -2$$
$$5x^3 - y^2 = y + 3$$

(5) Check if $(1, 1)$ solves.

$$3x - 5y = -2$$
$$x + 8y = 9$$

(6) Check if $(1, 1)$ solves.

$$x - y = 0$$
$$x + y = 2$$

(7) Check if $(1, 1)$ solves.

$$x - y = 0$$
$$x + y = 2$$
$$x + 5x = 3$$

(8) Check if $(1, 1)$ solves.

$$x - y = 0$$

(9) Check if $x = 1, y = 2, z = 3)$ solves.

$$x + y + 2z = 9$$
$$x - y - z = -4$$
$$3x + y - z = 2$$

(10) Check if $(\frac{1}{2}, \frac{5}{2})$ solves.

$$x - y = -2$$
$$x + y = 3$$

(11) Check if $(0, 0)$ solves.

$$x - 3y = 0$$
$$5x + y = 0$$

(12) Which (above) systems are linear?

8.2. 2×2 Linear by Graphing

"until one day... nothing happened"

Gameplan 8.2
(1) *Solve by Graphing*
(2) *Three Linear Cases*

2×2 By Graphing

Here, we consider systems with 2 linear equations and 2 unknowns or 2×2 *Linear Systems*, for short. Recall that the graph of a linear equation with two variables is a line. We solve the system by exploiting this fact. In other works, we graph one of the equations. All the point on this first graph satisfy the first equation. We then graph the second equation. The points on this graph satisfy the second equation. We then ask, which point satisfy both equations? The point that lie on both lines are the points that solve the system. Consider how many possible outcomes exist. One the two lines never meet. In this case, the system is called inconsistent. Moreover, *we will assume* that the only way for two lines to never meet is for them to be parallel.
The other extreme is when the two lines are identical, one on top of the other. In this case any point on one line is also on the other, thus any point on any of the lines is a solution to both equations, and thus a solution to the system. In this case, the equations are redundant since they describe the same points. The system is called *dependent*.
Finally, the only other option is the the two line meet exactly once. In this case, the system is called *consistent*. The only drawback to this method for solving the system is that it may sometimes be difficult to get a precise reading of the coordinates of the point of intersection.

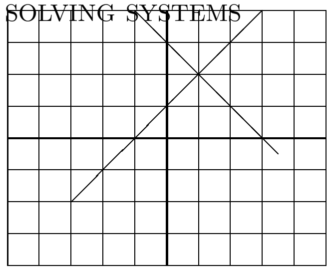
Examples

(1) Solve

$$y - x = 1$$
$$y + x = 3$$

Solution:

We graph the each of the equations on one graph and note the solution is inevitably revealed. The solution is $(1, 2)$.



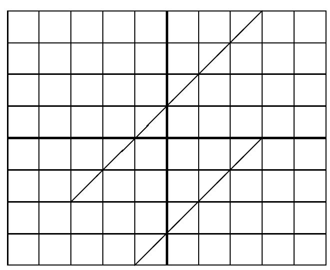
Just for fun we check our answer:
...on first equation we check: $2 - 1 = 1$ is true
...on second equation we check: $2 + 1 = 3$ also true! QED
(note QED is latin for "you can take this one to the bank")

(2) Solve

$$y - x = 1$$
$$y - x = -3$$

Solution:

We graph the each of the equations on one graph and note the solution is revealed with the greatest of ease.



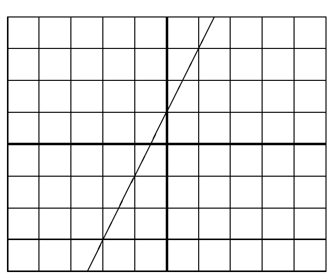
There is no solution. These lines are parallel. This system is *inconsistent*

(3) Solve

$$y - 2x = 1$$
$$3y - 6x = 3$$

Solution:

We graph.



The solution is immediately revealed. These equations represent the same points, thus the system is dependent. There are infinite many solution points. Any point on the first (on any) equation is a solution. Said another way we can eliminate one of these equations and the system is reduced to one equation, say $y - 2x = 1$. We offer several descriptions of the solution. The third one is preferred.

(a) All ordered pairs (x, y) such that $y - 2x = 1$.

(b) The set of all ordered pairs that of form $(x, 2x + 1)$ where x is any number

(c) $\{(a, 2a + 1) | a \in \mathbb{R}\}$

(4) Solve by graphing.

$$x + y = -2$$
$$x - y = -4$$

Solution:

We rewrite the equations in y-intercept form to make it easier to graph, then we graph them.

$$x + y = -2 \quad \rightarrow y = -x - 2$$
$$x - y = -4 \quad \rightarrow y = x + 4$$

We then graph each to obtain

We can now read the intersection point from the graph to obtain the expected solution $(-3, 1)$. Note, one drawback to the 'graphing' method is that we have to guess what the coordinates are. From the graph alone we don't know for example if the x coordinate is $x = -3$ or $x = -2.97$, etc..

On the other hand, the graphing method does provide an excellent way of understanding the various types linear systems. Consider the next example by graphing.

(5) Solve by graphing.

$$\begin{aligned} x+y&=-2\\ -x-y&=-4 \end{aligned}$$

Solution:

We rewrite the equations in y-intercept form to make it easier to graph, then we graph them.

$$\begin{aligned} x+y&=-2 && \rightarrow y=-x-2\\ x-y&=-4 && \rightarrow y=-x+4 \end{aligned}$$

We then graph each to obtain

This gives very clear picture of what happens. The lines never intersect in the real plane. Therefore, there is no real solution (x, y). This is a perfect example of an *inconsistent system.* The following example illustrates the possibility of infinite many solutions. Such a linear system is called a dependent system.

(6) Solve by graphing.

$$\begin{aligned} 2x+y&=-2\\ -4x-y&=-4 \end{aligned}$$

Solution:

We rewrite the equations in y-intercept form to make it easier to graph, then we graph them.

$$\begin{aligned} 2x + y &= -2 & &\rightarrow y = -2x - 2 \\ 4x + 2y &= -4 & &\rightarrow 2y = -4x - 4 \qquad \rightarrow y = -2x - 2 \end{aligned}$$

We then graph each to obtain

From the graph (or from the re-written equations) we see the two lines are overlapping everywhere. They are the same line, therefore any point on one line is also a point on the other line. The solution to the system is all points on the line. In other words, x could take on any value $x = a$, while $y = -2a - 2$. We say the system is dependent, or redundant and we say the solution set is

$$\text{all } x = a \qquad y = -2a - 2 \text{ for any } a$$

EXCERCISES 8.2

Solve by graphing.

(1)

$$\begin{aligned} 2x + y &= 3 \\ 4x + 10y &= 14 \end{aligned}$$

(2) $3x - 2y = -6$

(3)

$$\begin{aligned} 2x - 3y &= -1 \\ 4x - 10y &= -6 \end{aligned}$$

(4)

$$\begin{aligned} 2x - 3y &= -1 \\ 4x - 10y &= 5 \end{aligned}$$

(5)

$$\begin{aligned} 2x - 3y &= -1 \\ 4x - 10y &= -2 \end{aligned}$$

(6)

$$\begin{aligned} 2x - 3y &= 2 \\ 4x - 10y &= 4 \end{aligned}$$

(7)

$$\begin{aligned} 2x - 3y &= -3 \\ 4x - 10y &= -10 \end{aligned}$$

(8)

$$\begin{aligned} 2x - 3y &= -3 \\ 4x - 6y &= -6 \end{aligned}$$

(9)

$$\begin{aligned} 2x - 3y &= -3 \\ 4x - y &= 6 \end{aligned}$$

(10)

$$\begin{aligned} 2x + 3y &= 3 \\ -4x + y &= 4 \end{aligned}$$

(11)

$$\begin{aligned} 2x + 3y &= 3 \\ 2x + 3y &= 4 \end{aligned}$$

(12)

$$\begin{aligned} 2x + 3y &= 3 \\ 8x + 6y &= 6 \end{aligned}$$

(13)

$$\begin{aligned} 2x + 3y &= -1 \\ 8x - 6y &= 14 \end{aligned}$$

8.3. 2×2 Linear by Sub

"until one day... nothing happened"

Gameplan 8.3
(1) *Solve by Substitution*
(2) *Three Linear Cases*

2×2 By Substituting

One of the weakness of the of graphing method for solving systems is that it may be difficult to read the exact coordinates of the solution point. Using *Substitution* we can get exact solutions, overcoming this problem. The strategy is to take the information from one equation and transport it into the other equation, merging the two equations into one equation with one unknown. The rest is duck soup! Here is a sketch of the strategy.

(1) Pick one of the variables and one of the equations.
(2) Solve for the picked variable in the picked equation
(3) Substitute this variable into the other equation, eliminating its presence, and reducing the system into one equation one variable. Note the *picked* variable becomes the *eliminated* variable.
(4) solve this 1×1 system (if you can)*
(5) plug in this value into one of the original equations to recover the value of the *picked/eliminatied* variable.
(6) *If there are no solutions to this equation, the system is inconsistent. If there are infinite many solutions the system is dependent.

Examples

(1) Solve by substitution.
Solve

$$x + y = -2 \tag{8.1}$$
$$x - y = -4 \tag{8.2}$$

Solution:

The idea here is to take one of the variable in one of the equations, solve for it, and substitute into the other equation. In this case we will pick x on the second equation and solve for it. We will then substitute this into the first equation.

$$\begin{aligned} x - y &= -4 && \text{Given from (5.2) above} \\ x &= y - 4 && \text{CLA} \end{aligned}$$

We now substitute into (5.1), to obtain:

$$\begin{aligned} y - 4 + y &= -2 && \text{Sub into (5.1)} \\ 2y - 4 &= -2 && \text{BI} \\ 2y &= 2 && \text{CLA} \\ y &= 1. && \text{CLM} \end{aligned}$$

Now we can take $y = 1$ and substitute on either (5.1) or (5.2). We choose to substitute into (5.1) to obtain

$$\begin{aligned} x + 1 &= -2 && \text{Sub } y = 1 \text{ into (5.1)} \\ x &= -3. && \text{CLA} \end{aligned}$$

We conclude with the final solution

$$x = -3 \quad y = 1 \qquad \text{or simply} \qquad (-3, 1)$$

(2) Solve by Substitution

(1) $$2x + y = 3$$
(2) $$4x + 10y = 14$$

Solution:

(1) $$2x + y = 3 \qquad \text{given}$$
(2) $$4x + 10y = 14 \qquad \text{given}$$
(3) $$y = -2x + 3 \qquad \text{CLA on (1)}$$
(4) $$4x + 10(-2x + 3) = 14 \qquad \text{Sub(3) into (2)}$$
(5) $$4x + -20x + 30 = 14 \qquad \text{DL}$$
(6) $$-16x = 16 \qquad \text{CLA}$$
(7) $$x = -1 \qquad \text{CLM}$$
(8) $$y = -2(-1) + 3 \qquad \text{Sub (7) into (3)}$$
(9) $$y = 5 \qquad \text{BI}$$

(3) Solve by Substitution

$$\begin{aligned} y - 2x &= 1 \\ 3y - 6x &= 3 \end{aligned}$$

Solution:

(1)	$y - 2x = 1$	given
(2)	$3y - 6x = 3$	given
(3)	$y = 2x + 1$	CLA on(1)
(4)	$3(2x + 1) - 6x = 3$	Sub (3) onto (1)
(5)	$6x + 3 - 6x = 3$	DL
(6)	$3 = 3$	BI

Note that this equation is always true. Thus, the equations are independent, and the system can be reduced to one equation with two variables. For any value of a of x, if $x = a$ then $y - 2a = 1$ and the solution set can be described as $\{(a, 2a + 1) | a \in \mathbb{R}\}$

(4) Solve by Substitution

$$y - 2x = 1$$
$$3y - 6x = 5$$

Solution:

(1)	$y - 2x = 1$	given
(2)	$3y - 6x = 5$	given
(3)	$y = 2x + 1$	CLA on(1)
(4)	$3(2x + 1) - 6x = 5$	Sub (3) onto (1)
(5)	$6x + 3 - 6x = 5$	DL
(6)	$3 = 5$	BI

No matter what value x takes, equation (6) will never be true. Thus, it has no solution! Therefore we can conclude the system is inconsistent and has no solution.

EXCERCISES 8.3

Solve by substituting.

(1)
$$2x + y = 3$$
$$4x + 10y = 14$$

(2) $3x - 2y = -6$

(3)
$$2x - 3y = -1$$
$$4x - 10y = -6$$

(4)
$$2x - 3y = -1$$
$$4x - 10y = 5$$

(5)
$$2x - 3y = -1$$
$$4x - 10y = -2$$

(6)
$$2x - 3y = 2$$
$$4x - 10y = 4$$

(7)
$$2x - 3y = -3$$
$$4x - 10y = -10$$

(8)
$$2x - 3y = -3$$
$$4x - 6y = -6$$

(9)
$$2x - 3y = -3$$
$$4x - y = 6$$

(10)
$$2x + 3y = 3$$
$$-4x + y = 4$$

(11)
$$2x + 3y = 3$$
$$2x + 3y = 4$$

(12)
$$2x + 3y = 3$$
$$8x + 6y = 6$$

(13)
$$2x + 3y = -1$$
$$8x - 6y = 14$$

8.4. 2×2 Linear by killing a Variable

"until one day... nothing happened"

Gameplan 8.4
(1) *Solve by Eliminating a Variable*
(2) *Three Linear Cases*

2×2 Solve by Killing a Variable

The idea here is to add the two equations combining the information from each equation into one equation simultaneously killing one of the variables. If done correctly, this reduce a 2×2 system into a one by one. The rest is easy. However, we need a neat little theorem that proves we can add two equations together.

Add the Two Equations Theorem [ATE]

$$\text{if } a = b \text{ and } c = d \text{ then } a + c = b + d$$

(1)	Suppose $a = b$	is Given
(2)	and.. $c = d$	is also Given
(3)	then..$a + c = b + c$	By CLA on (1)
(4)	$c + b = d + b$	By CLA on (2)
(5)	$b + c = b + d$	By Commutative Law of Addition
(6)	$a + c = b + d$	Transitivity 3rd and 5th equations

All thanks to our beloved axioms!

Examples

(1) Solve:

$$x + y = -2$$
$$x - y = -4$$

Solution:

(1)	$x + y = -2$	Given
(2)	$x - y = -4$	Given
(3)	$x + x + y + -y = -2 + -4$	ATE (Kill y's)
(4)	$2x = -6$	B.I
(5)	$x = -3$	B.I.
(6)	$-3 + y = -2$	Sub (5) in (1)
	$y = 1$	CLA

Thus the solution is $(-3, 1)$ QED

(2) Solve:

$$2x + 3y = 8$$
$$5x - 2y = 1$$

Solution:

(1)	$2x + 3y = 8$	Given
(2)	$5x - 2y = 1$	Given
(3)	$10x + 15y = 40$	CLM on (1)
(4)	$-10x + 4y = -2$	CLM on (2) (getting ready to kill the x's)
(5)	$10x - 10x + 15y + 4y = 40 - 2$	ATE (Kill x's)
(6)	$19y = 38$	B.I
(7)	$y = 2$	B.I.
(8)	$2x + 3(2) = 8$	Sub (7) in (1)
(9)	$2x = 2$	CLA
(10)	$x = 1$	CLM

Thus, the solution is $(1, 2)$ QED

(3) Solve:

$$x + y = -2$$
$$3x + 3y = -6$$

Solution:

(1)	$x + y = -2$	Given
(2)	$3x + 3y = -6$	Given
(3)	$-3x - 3y = 6$	CLM on (1)(preparing to kill x's)
(4)	$3x + -3x + 3y + -3y = 6 - 6$	ATE
(5)	$0 = 0$	B.I.

Since equation (5) is always true, we conclude that the equations are dependent and redundant. The solution is all points of the form $x = aything$ and $y = -x - 2$ or said another way $\{(a, -a - 2) | a \in \mathbb{R}\}$

(4) Solve by Substitution

$$y - 2x = 4$$
$$3y - 6x = 3$$

Solution:

We begin with the killing mentality on y. In order to eliminate y we need to coefficients to become additive inverses. Since the second equation has a 3 coefficient, we will strive to get a -3 on the y-coefficient on the first equation. We do this by multiplying the entire first equation by a -3. The resulting system is given by.. CLM on first equation, then..

$$-3y + 6x = -12$$
$$3y - 6x = 3$$

We now add the two equations by ATE

$$0 = -9$$

We now pause to reason through this. We are looking for a solution pair (x, y) that satisfies the initial system. The existence of such numbers implies that there are real numbers (x, y) that make the statement '$0 = 9$' true. Obviously this is impossible, which leads to the conclusion that no such x and y exist, thus the system is inconsistent. By using the graphing method it is easy to recognize an inconsistent system when one sees one, namely parallel lines that never meet. Here we have a perfect 'algebraic' picture of what an inconsistent system looks like, namely and absurd statement such as "$0 = 9$".

EXCERCISES 8.4

Solve by Killing a variable.

(1)
$$\begin{aligned} 2x + y &= 3 \\ 4x + 10y &= 14 \end{aligned}$$

(2)
$$\begin{aligned} 2x - 3y &= -1 \\ 4x - 10y &= -6 \end{aligned}$$

(3)
$$\begin{aligned} 2x - 3y &= -1 \\ 4x - 10y &= 5 \end{aligned}$$

(4)
$$\begin{aligned} 2x - 3y &= -1 \\ 4x - 10y &= -2 \end{aligned}$$

(5)
$$\begin{aligned} 2x - 3y &= 2 \\ 4x - 10y &= 5 \end{aligned}$$

(6)
$$\begin{aligned} 2x - 3y &= -3 \\ 4x - 10y &= -10 \end{aligned}$$

(7)
$$\begin{aligned} 2x - 3y &= -1 \\ 4x - y &= -6 \end{aligned}$$

(8)
$$\begin{aligned} 2x - 3y &= -3 \\ 4x - y &= 6 \end{aligned}$$

(9)
$$\begin{aligned} 2x + 3y &= 3 \\ -4x + y &= 4 \end{aligned}$$

(10)
$$\begin{aligned} 2x + 3y &= 3 \\ 2x + 5y &= 4 \end{aligned}$$

(11)
$$\begin{aligned} 2x + 3y &= 3 \\ 8x + 6y &= 6 \end{aligned}$$

(12)
$$\begin{aligned} 2x + 3y &= -1 \\ 8x - 6y &= 14 \end{aligned}$$

8.5. 2×2 Linear by Cramer

"until one day... nothing happened"

Gameplan 8.5
(1) *A 2×2 Matrix*
(2) *Determinants*
(3) *Cramers*

2×2 Matrices

Simply put, we can define a 2×2 matrix as 'table' of values with two rows of entries and two column. The following are excellent examples of matrices

$$\begin{pmatrix} 2 & 3 \\ -5 & 7 \end{pmatrix} \qquad \begin{pmatrix} 7 & \sqrt{3} \\ 4 & 2 \end{pmatrix}$$

Pause for a second and make a mental note of all the families we've met thus far. We have met the family of *Natural Numbers,* $\mathbb{N}$, then the integers $\mathbb{Z}$, rationales $\mathbb{Q}$, reals $\mathbb{R}$, complex $\mathbb{C}$, and polynomials $\mathbb{R}[x]$. In each instance, we entertained questions regarding the major binary operations, how to add, multiply, subtract and divide within each family. We now meet the family of 2×2 matrices. The next natural question is "how do we add matrices, multiply matrices, subtract or divide matrices?"
Unfortunately, we will not resolve these questions in this class. However, further studies in mathematics will surely get you there, and you will know how to add, multiply, and subtract matrices. We will however entertain one questions, *how do we take the 'absolute value' of a matrix?* The answer is we don't. Our matrices will not have an 'absolute value' perse, yet they will have something resembling it called its *determinant.* A couple of things should be immediately cleared up, one how to write 'the determinant of a matrix' and two how to calculate it.
Let us take the first question. Suppose A is a Matrix given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Each of the following means the same quantity, namely 'the determinant of A'

$$det(A), \qquad |A|, \qquad \begin{vmatrix} a & b \\ d & e \end{vmatrix}, \qquad det \begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

The next question is how do we define the determinant of a two by two matrix. The first time you see the definition of a matrix it may feel a bit unnatural, don't worry it is not you. The definition comes from afar, and you will need to be patient to understand why the definition was cooked up the way it was cooked up. Nevertheless, the determinant is not difficult to calculate. We will first define it, then follow the definition with a bunch of examples.

2×2 Determinants Defined

$$det \begin{pmatrix} a & b \\ d & e \end{pmatrix} = ae - bd$$

2 × 2 Determinant Examples

(1) Find

$$det \begin{pmatrix} 3 & 5 \\ 6 & 2 \end{pmatrix}$$

Solution:

$$\begin{aligned} det \begin{pmatrix} 3 & 5 \\ 6 & 2 \end{pmatrix} &= 3 \cdot 2 - 5 \cdot 6 && \text{Def Det} \\ &= 6 - 30 && \text{TT} \\ &= -24 && \text{BI} \end{aligned}$$

(2) Find

$$det \begin{pmatrix} 3 & {}^-5 \\ 6 & 2 \end{pmatrix}$$

Solution:

$$\begin{aligned} det \begin{pmatrix} 3 & {}^-5 \\ 6 & 2 \end{pmatrix} &= 3 \cdot 2 -^- 5 \cdot 6 && \text{Def Det} \\ &= 6 + 30 && \text{TT} \\ &= 36 && \text{BI} \end{aligned}$$

(3) Find

$$det \begin{pmatrix} {}^-2 & 5 \\ {}^-3 & 3/2 \end{pmatrix}$$

Solution:

$$\begin{aligned} det \begin{pmatrix} {}^-2 & 5 \\ {}^-3 & 3/2 \end{pmatrix} &=^- 2 \cdot \frac{3}{2} -^- 3 \cdot 5 && \text{Def Det} \\ &= -3 + 15 && \text{BI} \\ &= 12 && \text{BI} \end{aligned}$$

$$\begin{aligned} det\begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix} &= 2\cdot 1 - 3\cdot 5 && \text{Def Det} \\ &= 2 - 15 && \text{BI} \\ &= -13 && \text{BI} \end{aligned}$$

CRAMERS RULE

Cramers Rule for a 2 × 2 system solves the general consistent system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

Of course we could also solve the system either by substitution or adding the equations. In either case, the solution is obtained is

$$x = \frac{ce - bf}{\mathbf{ae} - \mathbf{bd}} \qquad y = \frac{af - dc}{\mathbf{ae} - \mathbf{bd}}$$

Cramers' brilliant idea is to note the solution for x and y is each made up of *determinants*! Note the denominators in x and y are both the same, $ae - bd$ and if we go back to our system, we find a matrix from the coefficients.

$$\begin{aligned} \mathbf{a}x + \mathbf{b}y &= c \\ \mathbf{d}x + \mathbf{e}y &= f, \end{aligned}$$

In other words if we take the matrix given by the coefficients on the left side of the system, then find its determinant, then we have the denominators for both the x value and the y value.

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{d} & \mathbf{e} \end{vmatrix} = \mathbf{ae} - \mathbf{bd}$$

Now that we know the denominators if we just figure out how to get each of the numerators of x and y we will always know how to completely determine the solution to a consistent linear two by two system. Consider the numerator for the y value.

$$y = \frac{\mathbf{af} - \mathbf{dc}}{ae - bd}$$

This is also the determinant of a matrix. To obtain this matrix we simply 'replace' the y column with the right-side column. In other words, we start with the same matrix used for the denominator, namely

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{d} & \mathbf{e} \end{pmatrix}$$

We then replace the y column $\begin{bmatrix} \mathbf{b} \\ \mathbf{e} \end{bmatrix}$ with the right-side column $\begin{bmatrix} c \\ f \end{bmatrix}$ to obtain then numerator matrix

$$\begin{pmatrix} \mathbf{a} & c \\ \mathbf{d} & f \end{pmatrix}$$

We take the determinant of this matrix, and obtain the numerator for the y value of our solution.

$$y = \frac{\begin{vmatrix} \mathbf{a} & c \\ \mathbf{d} & f \end{vmatrix}}{\begin{vmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{d} & \mathbf{e} \end{vmatrix}} = \frac{af - dc}{ae - db}$$

Similarly to find the x value, we first start with the denomiantor matrix, $\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{d} & \mathbf{e} \end{pmatrix}$ then substitute the x-column $\begin{bmatrix} \mathbf{a} \\ \mathbf{d} \end{bmatrix}$ with the right-side column $\begin{bmatrix} c \\ f \end{bmatrix}$ to obtain the numerator matrix for x, giving us the precise value for x as,

$$x = \frac{\begin{vmatrix} c & \mathbf{b} \\ f & \mathbf{e} \end{vmatrix}}{\begin{vmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{d} & \mathbf{e} \end{vmatrix}} = \frac{ce - bf}{ae - db}$$

In general the solution will be a determinant fraction. The denominator will always have and x-column a y-column, the numerator will be identical except for one of the columns: if we are solving for x we replace the x column with the right-side column, if we want to solve for y we replace the y column with the right-side column, etc.

Examples by Cramers Rule

(1) Solve

$$x + 3y = -2$$
$$5x - y = 6$$

Solution:

$$x = \frac{\begin{vmatrix} -2 & \mathbf{3} \\ 6 & \mathbf{-1} \end{vmatrix}}{\begin{vmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{5} & \mathbf{-1} \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} \mathbf{1} & -2 \\ \mathbf{5} & 6 \end{vmatrix}}{\begin{vmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{5} & \mathbf{-1} \end{vmatrix}}$$

We calculate each determinant to conclude

$$x = \frac{2 - 18}{-1 - 15} = 1 \qquad y = \frac{6 + 10}{-1 - 15} = -1$$

(2) Solve

$$3x + y = -2$$
$$x - 2y = -4$$

Solution:

$$x = \frac{\begin{vmatrix} -2 & 1 \\ -4 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 3 & -2 \\ 1 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix}}$$

We calculate each determinant to conclude

$$x = \frac{4+4}{-6-1} = -\frac{8}{7} \qquad y = \frac{-12+2}{-6-1} = \frac{10}{7}$$

(3) Solve

$$3x + y = -2$$
$$6x + 2y = -4$$

Solution:

$$x = \frac{\begin{vmatrix} -2 & 1 \\ -4 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix}}$$

We calculate each determinant to obtain

$$x = \frac{-4+4}{6-6} = \frac{0}{0} \qquad y = \frac{-12+12}{6-6} = \frac{0}{0}.$$

These solutions pose a problem since the quantity $\frac{0}{0}$ is not real numbers. In this case, Cramers Rule does not help us solve the system. Moreover, this example makes it clear that Cramers rule applies only to consistent systems where $ae - bd \neq 0$. These systems are either dependent or inconsistent, and it may be better to apply the substitution method or the 'adding equations' method.

EXCERCISES 8.5

Solve using Cramers Rule (if possible, otherwise use a different method to solve).

(1)

$$2x + y = 3$$
$$4x + 10y = 14$$

(2)

$$2x - 3y = -1$$
$$4x - 10y = -6$$

(3)

$$2x - 3y = -1$$
$$4x - 10y = 5$$

(4)

$$2x - 3y = -1$$
$$4x - 10y = -2$$

(5)

$$2x - 3y = 2$$
$$4x - 10y = 5$$

(6)

$$2x - 3y = -3$$
$$4x - 10y = -10$$

(7)

$$\begin{aligned} 2x - 3y &= -1 \\ 4x - y &= -6 \end{aligned}$$

(8)

$$\begin{aligned} 2x - 3y &= -3 \\ 4x - y &= 6 \end{aligned}$$

(9)

$$\begin{aligned} 2x + 3y &= 3 \\ -4x + y &= 4 \end{aligned}$$

(10)

$$\begin{aligned} 2x + 3y &= 3 \\ 2x + 5y &= 4 \end{aligned}$$

(11)

$$\begin{aligned} 2x + 3y &= 3 \\ 8x + 6y &= 6 \end{aligned}$$

(12)

$$\begin{aligned} 2x + 3y &= -1 \\ 8x - 6y &= 14 \end{aligned}$$

8.6. 3×3 Linear

3×3 Systems

My favorite strategy for 3×3 linear systems is the generalization of the previous method. We will begin by deciding on which variable we will kill first. We then combine the equations in pairs. This should reduce our system from a $3\times$ to a 2×2 system. We kill another variable and reduce to a 1×1 system. The rest is Duck Soup or Piece-a-Cake!!

Examples

(1)

$$x + 2y + z = 8 \tag{1}$$
$$-x - y + z = 0 \tag{2}$$
$$x + 2y - 3z = -4 \tag{3}$$

Solution:

$$\begin{array}{ccc} x + 2y + z = 8 & & \\ & y + 2z = 8 & \\ -x - y + z = 0 & & 2y = 4 \\ & y - 2z = -4 & \\ x + 2y - 3z = -4 & & \end{array}$$

Note, by second column all x's have been killed and we've reduced the system to a 2×2 system. By third column all z's have been eliminated, and we're down to a 1×1 system. Once the system is reduced to a 1×1 system it is easy to solve. We get that $y = 2$ then by substituting in previous equations we get that $x = 1$ and $z = 3$ or for short $(1, 2, 3)$. This system was particularly easy to solve because the equations were ready to be added to eliminate the appropriate variables. Sometimes you may need to use CLM first before the variables are ready to be eliminated.

(2)

$$3x - 2y + 2z = 1 \tag{1}$$
$$-2x - y + 5z = -7 \tag{2}$$
$$x + 2y - 3z = 4 \tag{3}$$

Solution:

This time we are going to kill the x's first, then by the third column we will kill the y's. Finally, solve for z and start plugging back into previous equation.

$$\begin{aligned} 3x-2y+2z&=1\\ -2x-y+5z&=-7\\ x+2y-3z&=4 \end{aligned} \quad \begin{aligned} 6x-4y+4z&=2\\ -6x-3y+15z&=-21\\ 6x+12y-18z&=24 \end{aligned} \quad \begin{aligned} -7y+19z&=-19\\ 9y-3z&=3 \end{aligned} \quad \begin{aligned} -63y+171z&=-171\\ 63y-21z&=21 \end{aligned} \quad 150z=150$$

The rest is Duck Soup! by substituting back into previous equations we get the solution $(1,0,-1)$

EXCERCISES 8.6

(1)

$$\begin{aligned} 2x-3y+5z&=4\\ -2x-5y+6z&=-1\\ x+4y+2z&=7 \end{aligned}$$

(2)

$$\begin{aligned} -2x-3y+5z&=3x-4y+z\\ -2x-5y+6z&=-1\\ 2x+4y+2z&=8 \end{aligned}$$

(3)

$$\begin{aligned} -2x-3y+5z&=0\\ -2x-5y+8z&=1 \end{aligned}$$

(4)

$$\begin{aligned} -2x-3y+5z&=0\\ -2x-5y+8z&=1\\ 4x+6y-10&=0 \end{aligned}$$

(5)

$$\begin{aligned} -2x-3y+5z&=0\\ -2x-5y+8z&=1\\ 4x+6y-10&=1 \end{aligned}$$

(6)

$$\begin{aligned} 3x+3y+5z&=10\\ -2x-2y+8z&=16\\ 4x+6y-z&=-4 \end{aligned}$$

(7)

$$\begin{aligned} 3x+3y+4z&=8\\ -2x-2y+8z&=16\\ 4x+6y-5z&=-12 \end{aligned}$$

(8)

$$\begin{aligned} 3x-3y+4z&=8\\ -2x-y+8z&=16\\ 6x+6y-5z&=-10 \end{aligned}$$

(9)

$$\begin{aligned} 3x-3y+4z&=1\\ -2x-y+8z&=7\\ 6x+6y-5z&=1 \end{aligned}$$

(10)

$$\begin{aligned} 3x-3y+4z&=1\\ -2x-y+8z&=7\\ 6x-6y+8z&=-1 \end{aligned}$$

(11)

$$\begin{aligned} 3x-3y+4z&=1\\ -2x-y+8z&=7\\ 6x+6y+8z&=2 \end{aligned}$$

8.7. Non-Linear

Non-Linear Systems

Unlike linear systems, non-linear systems may have 0, 1, 2, 3 or any number of solutions. Most methods we used in the previous section still apply here. Cramers rule will not help here. Graphing and substitution are specially appealing. For example, by graphing the equations $3x + 2x = 6$ and the equation $x^2 + y^2 = 25$ you may gain some insight into the possible number of solutions. One graph is a line the other a circle. How could two graphs intersect? A moments' thought reveals that you may have 0 1, or two solutions to such a system, where one equation is a line and the other a circle.

Examples

(1) Solve

$$\begin{aligned} x^2 + y^2 &= 16 \\ y &= -2x + 2 \end{aligned}$$

Solution:

We graph both solutions and quickly appreciate the strengths and weaknesses of of the graphing method. It is nice to see the solutions quickly and effortlessly. However, we can only guess at their exact coordinates. The two solutions are approximately

$$(2.7, -3) \text{ and } (-1, 3.8)$$

This brings us to the next method, substitution. Using substitution we should get more precise results.

(2) Solve

$$\begin{aligned} x^2 + y^2 &= 16 \\ y &= -2x + 2 \end{aligned}$$

Solution:

We substitute the second equation into the first, to obtain

$$\begin{aligned}
x^2 + (-2x+2)^2 &= 16 && \text{Substitute}\\
x^2 + 4x^2 - 8x + 4 &= 16 && \text{Foil}\\
5x^2 - 8x - 12 &= 0 && \text{simplify}\\
x &= \frac{8 \pm \sqrt{64+240}}{10} && \text{QF}\\
x \approx 2.54 \text{ or } x &\approx -.94 && \text{Calculator}
\end{aligned}$$

We substitute each of these into the linear equation to obtain the corresponding y. We see that the solutions are However, we can only guess at their exact coordinates. The two solutions are approximately

$$(2.54, -3.09) \text{ and } (-.94, 3.89)$$

(3) Solve

$$\begin{aligned}
x^2 + y^2 &= 16\\
\frac{x^2}{4} + \frac{y^2}{36} &= 1
\end{aligned}$$

Solution:

We will multiply the second equation by -4 yielding a $-x^2$ term on the second equation, adding it to the first equation will eliminate the x's.

$\begin{matrix} x^2+y^2=16 \\ \frac{x^2}{4}+\frac{y^2}{36}=1 \end{matrix}$ becomes $\begin{matrix} x^2+y^2=16 \\ -x^2-\frac{y^2}{9}=-4 \end{matrix}$ adding... $y^2 - \frac{y^2}{9} = 12$

Now, that x has been eliminated, we solve for y.

$$\begin{aligned}
y^2 - \frac{y^2}{9} &= 12 && \text{Above}\\
\frac{8y^2}{9} &= 12 && \text{BI}\\
y^2 &= \frac{27}{2} && \text{BI}\\
y &= \pm\sqrt{27/2} && \text{SRP}
\end{aligned}$$

We now substitute each of these values of y into one of the previous equations to find the corresponding x values.

$$
\begin{aligned}
x^2 + y^2 &= 16 && \text{given} \\
x^2 + 27/2 &= 16 && \text{Substitute } y = \sqrt{27/2} \\
x^2 &= 16 - 27/2 && \text{CLA} \\
x^2 &= 5/2 && \text{BI} \\
x &= \pm\sqrt{5/2} && \text{SRP}
\end{aligned}
$$

This yields two solutions $(\sqrt{5/2}, \sqrt{27/2})$ and $(-\sqrt{5/2}, \sqrt{27/2})$. We move on to the other y value. Substitute $y = -\sqrt{27/2}$ and get two more points, $(\sqrt{5/2}, -\sqrt{27/2})$ and $(-\sqrt{5/2}, -\sqrt{27/2})$. A graph of the system confirms there are a total of 4 solutions, and we have them all.

(4) Solve

$$
\begin{aligned}
x^2 + y^2 &= 4 \\
x - y &= 0
\end{aligned}
$$

Solution:

$$
\begin{aligned}
&(1) & x^2 + y^2 &= 4 && \text{given} \\
&(2) & x - y &= 0 && \text{given} \\
&(3) & x &= y && \text{CLA} \\
&(4) & y^2 + y^2 &= 4 && \text{Sub (3) into (1)} \\
&(4) & 2y^2 &= 4 && \text{BI} \\
&(4) & y^2 &= 2 && \text{CLM} \\
&(4) & y &= \pm\sqrt{2} && \text{SRP}
\end{aligned}
$$

We first consider the case that $y = \sqrt{2}$ and plug into equation (3) and get that $x = \sqrt{2}$, thus we get our first solution $(\sqrt{2}, \sqrt{2})$. Second, we condider the other value of $y = -\sqrt{2}$ to get the other solution $(-\sqrt{2}, -\sqrt{2})$. A graph of the equations confirms this.

(5) Solve

$$x^2 + y^2 = 4$$
$$(x-2)^2 + y^2 = 1$$

Solution:

(1)	$x^2 + y^2 = 4$	given
(2)	$(x-2)^2 + y^2 = 1$	given
(3)	$-x^2 - y^2 = -4$	CLM on (1)
(4)	$(x-2)^2 - x^2 = -3$	ATET add(2) and (3) kill y's
(5)	$x^2 - 4x + 4 - x^2 = -3$	FOIL
(6)	$4x = 7$	BI
(7)	$x = \frac{7}{4}$	CLM
(8)	$\left(\frac{7}{4}\right)^2 + y^2 = 4$	Sub into (1)
(9)	$\frac{49}{16} + y^2 = 4$	BI
(10)	$y^2 = 4 - \frac{49}{16}$	CLA
(11)	$y^2 = \frac{64}{16} - \frac{49}{16}$	BI
(12)	$y^2 = \frac{15}{16}$	ATT
(13)	$y = \pm\sqrt{\frac{15}{16}}$	ATT

thus we get two solution points...$(\frac{7}{4}, \sqrt{\frac{15}{16}})$ and $(\frac{7}{4}, -\sqrt{\frac{15}{16}})$

EXCERCISES 8.7

(1)

$$x^2 + y^2 = 4$$
$$y - x^2 = 0$$

(2)

$$x^2 + y^2 = 4$$
$$(x-3)^2 + y^2 = 1$$

(3)

$$x^2 + y^2 = 4$$
$$-y - x^2 = 0$$

(4)

$$x^2 + y^2 = 4$$
$$-y^2 - x = 0$$

(5)

$$x^2 + y^2 = 4$$
$$y - \sqrt{2} = -1(x - \sqrt{2})$$

(6)

$$x^2 + y^2 = 4$$
$$y - \sqrt{2} = -1(x - \sqrt{2})$$

(7)

$$x^2 + y^2 = 4$$
$$-y + x = 0$$

(8)

$$x^2 + y^2 = 1$$
$$x^2 - y^2 = 1$$

(9)

$$x^2 + y^2 = 5$$
$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

(10)

$$x^2 - y = 0$$
$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

8.8. CHAPTER review

CHAPTER REVIEW

The Strategies:

(1) *Graphing*
   (a) Simple. Graph each of the equations and find the points of intersection.
   (b) The good thing is that it gives you great insight into the number of soluitons and a rough estimate of courdinates.
   (c) The bad thing is that often the estimate can not be made precise.
(2) *Substituting*
   (a) Pick one of the variables and one of the equations.
   (b) Solve for the picked variable in the picked equation
   (c) Substitute this variable into the other equation, eliminating its presence, and reducing the system into one equation one variable. Note the *picked* variable becomes the *eliminated* variable.
   (d) solve this 1×1 system (if you can)*
   (e) plug in this value into one of the original equations to recover the value of the *picked/eliminatied* variable.
   (f) *If there are no solutions to this equation, the system is inconsistent. If there are infinite many solutions the system is dependent.
(3) *Eliminating a Variable*
   (a) pick a variable to eliminate
   (b) combine the equations by pairs until you reduce the system by one equation and by one variable.
   (c) Repeat until you have a 1×1 system.
   (d) the procedure may be a bit different if the system is inconsistent or dependent.

Excercises 8.8:

(1)
$$\begin{aligned} x + y &= 7 \\ -x + 5y &= 5 \end{aligned}$$

(2)
$$\begin{aligned} -2x + 2y &= 7 \\ -3x + 5y &= 5 \end{aligned}$$

(3)
$$\begin{aligned} 2x - 5y &= 7 \\ -x + 5y &= 5 \end{aligned}$$

(4)
$$\begin{aligned} 2x - 5y &= 7 \\ -2x + 5y &= 5 \end{aligned}$$

(5)
$$\begin{aligned} x + y + z &= 3 \\ -x + 5y - 3z &= 1 \\ 3x - 2y + 2z &= 3 \end{aligned}$$

(6)
$$\begin{aligned} x^2 + y^2 &= 1 \\ y^2 + (x-4)^2 &= 4 \end{aligned}$$

(7)

$$\begin{aligned} x + y + z &= 3 \\ -x + 5y - 3z &= 1 \\ 3x - 15y + 9z &= -3 \end{aligned}$$

(8)

$$\begin{aligned} 2x - 5y &= 7 \\ -2x + 5y &= -7 \end{aligned}$$

(9)

$$\begin{aligned} x^2 - y^2 &= 1 \\ 10y &= x^2 \end{aligned}$$

(10)

$$\begin{aligned} x^2 + y^2 &= 1 \\ y^2 + (x-4)^2 &= 9 \end{aligned}$$

(11)

$$\begin{aligned} x^2 + y^2 &= 9 \\ y &= x^2 \end{aligned}$$

(12)

$$\begin{aligned} x^2 + y^2 &= 9 \\ \frac{y^2}{4} + \frac{x^2}{36} &= 1 \end{aligned}$$

CHAPTER 9

Geometry

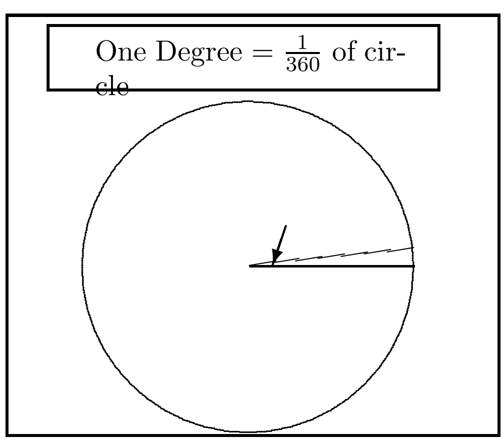
9.1. Angles

"until one day... nothing happened"

Gameplan 9.1
(1) *One Degree*
(2) *One Radian*
(3) *Converting*

One Degree

Degrees are units of measurement for angles. A one-degree angle is formed by slicing a circle into 360 equal pieces.

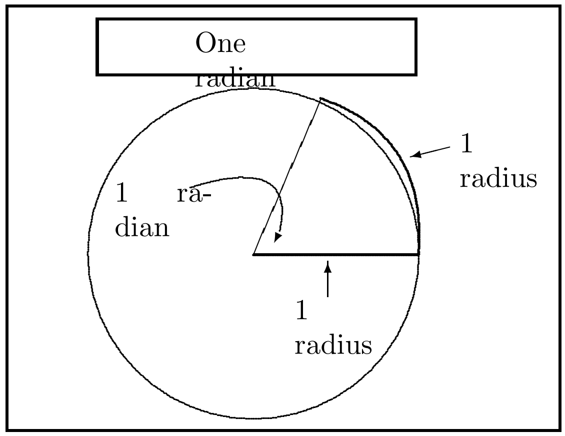


This means around the entire circle there are a total of 360 degrees. Furthermore, half the circle will create an angle measuring a total of 180°. Half of that would yield 90°. Consider the following famous angles. See the following graphs for a look at some of the most famous angles given in degrees.

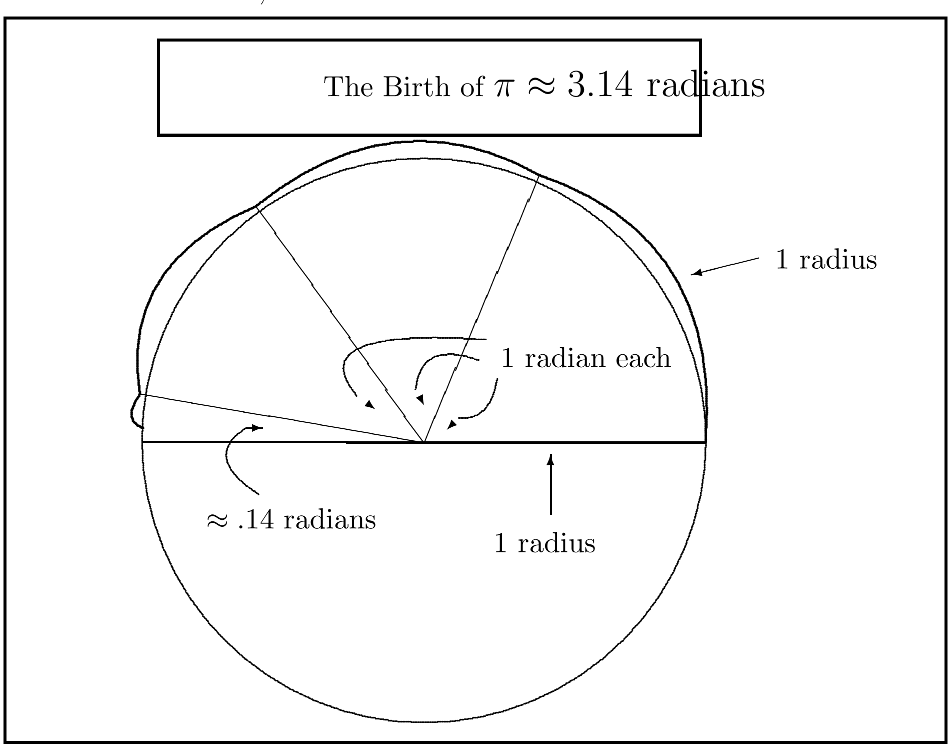
The idea of braking up the circle into 360 pieces to create *a degree* begs the question, *why 360?* It could have just as easily been 100 pieces, so that each 'degree' would constitute 1/100 of the circle. Indeed, there are other units to measure angles which do not rely on degrees. The competing units of measurement are *radians.* To fully understand radians, it is essential to understand *one radian.*

One Radian

One Radian is the angle obtained by taking the length of the radius of a circle over the arc-length.



Moreover, with the invention of *one radian* came the birth of one of the most celebrated numbers of all times, π. *Pi* is the exact number of radians that fit in a semicircle. Observe,



Here are the radian versions of famous angles. Compare these with the degree versions.

Note we will take the default units of angles to be radians. In other words, we will often speak of angles that measure $\frac{\pi}{3}$. For starters, we can easily approximate the quantity $\frac{\pi}{3}$ using a decimal approximation, as

$$\frac{\pi}{3} \approx \frac{3.14}{3} \approx= 1.046$$

but the question remains, 1.046 what? 1.046 radians or 1.046 degrees? In cases like this we will always mean 1.046 *radians,* the default units.

Converting

We will begin with a common sense approach to the conversion, which we will then refine into a more formal process. Begin by considering the the conversion of 180° into radians. By definition, *one degree* is 1/360th of a circle, thus 180° make up exactly half a circle. Meanwhile, the number of *radians* in half a circle is by definition π radians, thus 180° converts to exactly π radians.

We can cut the 180 degrees into 3 equal pieces to obtain angles of $60°$ each. The equivalent in radians would be to cut π radians into 3 equal pieces to yield angles of exactly $\pi/3$ radians each. It follows that $60° = \pi/3$ while $120° = 2\pi/3$, etc..

Similarly if we cut the semicircle into 4 equal pieces, each piece would yield an angle of $45°$ or in radians $\pi/4$

Now we refine our converting technique. The following is based strictly on the way we defined a degree and a radian. Recall one degree is the angle obtained by taking 1/360th of a circle. Said another way, $180°$ make up half a circle. While π radians also make up half a circle. Therefore we conclude the following measurement to be equal,

$$180° = \pi rad$$

Then,

$$\begin{aligned} 180deg &= \pi rad && \text{def of each} \\ deg &= \frac{\pi}{180} rad && \text{CLM} \end{aligned}$$

Similarly, we can conclude;

$$\begin{aligned} 180deg &= \pi rad && \text{def of each} \\ \frac{180}{\pi} deg &= rad && \text{CLM} \end{aligned}$$

The conclusion is excellent. The quantity deg or $^\circ$ can be treated as any other algebraic quantity. In particular we can substitute $\frac{\pi}{180} rad$ for deg at our leasure. We will call this the *Degree Radians Conversion Theorem* [DRC].

(1) We begin with a modest example, converting $137deg$ to radians.

Solution:

$$\begin{aligned} 137deg &= 137 \left(\frac{\pi}{180} rad \right) && \text{DRC} \\ &= \frac{137\pi}{180} rad && \text{BI} \\ &\approx 2.39 rad && \text{CALC} \end{aligned}$$

(2) Now we convert $2rad$ to degrees.

Solution:

$$\begin{aligned} 2rad &= 2 \left(\frac{180}{\pi} deg \right) && \text{DRC} \\ &= \frac{360}{\pi} rad && \text{BI} \\ &\approx 114.65 deg && \text{CALC} \end{aligned}$$

(3) Now we convert $\frac{3\pi}{5}$ to degrees.

Solution:

$$\begin{aligned}
\frac{3\pi}{5} &= \frac{3\pi}{5} rad && \text{Default Units} \\
&= \frac{3\pi}{5}\left(\frac{180}{\pi} deg\right) && \text{DRC} \\
&= 108 deg && \text{noddle power}
\end{aligned}$$

(4) Now we convert 90° to radians.

Solution:

$$\begin{aligned}
90^\circ &= 90\left(\frac{\pi}{180} rad\right) && \text{DRC} \\
&= \frac{90\pi}{180} rad && \text{BI} \\
&= \frac{\pi}{2} rad && \text{BI}
\end{aligned}$$

EXCERCISES 9.1

(1) draw angle $\frac{\pi}{6}$
(2) draw angle $\frac{\pi}{7}$
(3) draw angle 210°
(4) Convert 90deg
(5) Convert 30°
(6) Convert 210deg
(7) Convert 330°
(8) Convert 135deg
(9) Convert 20°
(10) Convert 60deg
(11) Convert 660°
(12) Convert 360deg
(13) Convert $\frac{\pi}{3}$rad
(14) Convert $\frac{2\pi}{3}$rad
(15) Convert 1rad
(16) Convert 2rad
(17) Convert 3rad
(18) Convert $\frac{3\pi}{2}$
(19) Convert $\frac{7\pi}{4}$
(20) Convert $\frac{13\pi}{4}$

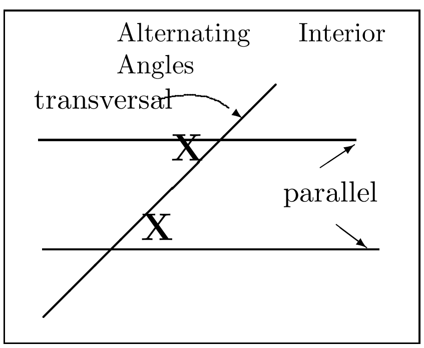
9.2. Radians & Degrees

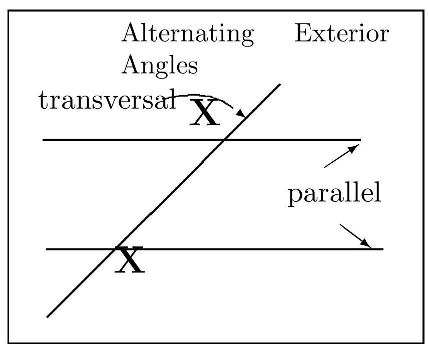
"until one day... nothing happened"

Gameplan 9.2
(1) *Famous Angles*
(2) *VAT*
(3) *PAL*

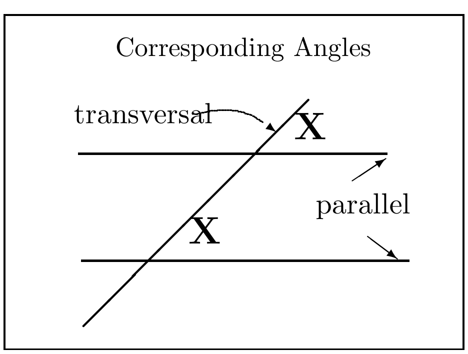
Famous Angles

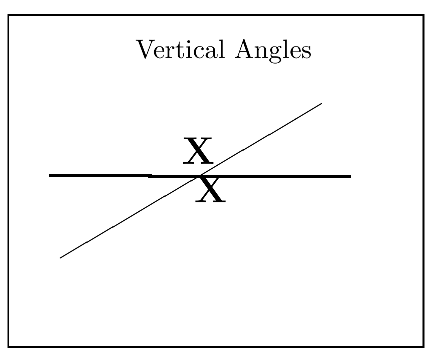
A line intersecting two parallel lines is often referred to as a *transversal line.* The intersection of a transversal with two parallel lines creates several angles, each with a special name. The following is a concise debriefing of some of these angles.
Notice on the first illustration, the angles stay inside the parallel lines, thus 'interior' while they lie on alternating sides of the transversal, thus the name 'alternating interior. There are two pairs of alternating interior angles. Similarly, there are two pairs of 'alternating exterior' angles.



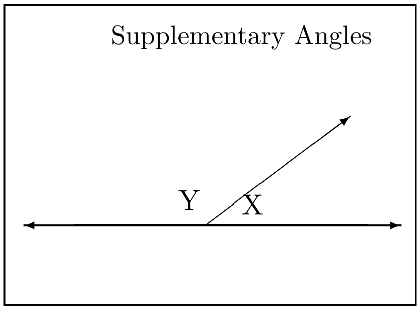


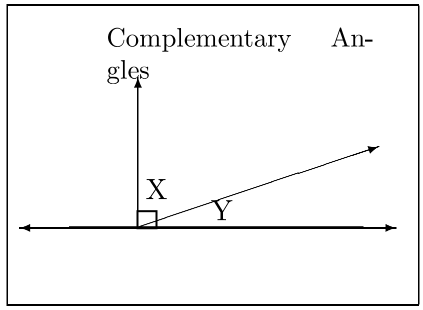
Corresponding angles remain on the same side of the transversal, one inside the parallel lines and one outside the parallel lines.





Note, *vertical angles* are defined by the intersection of any two distinct lines.
Suppose a segment goes through trough points AB, and suppose point A is on some other line. The resulting pair of angles depicted below are called supplementary angles. Said another way, supplementary angles are pairs of angles whose measurements sum to 180°.



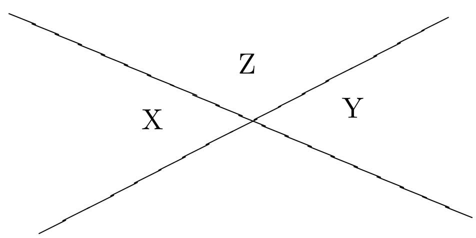


Vertical Angles Theorem [VAT]

Angles with the same measurement are called *congruent angles.* The *vertical angles theorem [VAT]* says that if angle X and angle Y are vertical, then they are congruent. We will denote 'angle Y is congruent to angle X' by

$$X \cong Y$$

Suppose angle X and Y are given to be vertical angles, then:

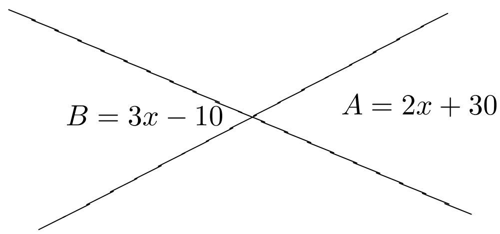


Sketch of Proof:

$X + Z = 180^\circ$	Supplementary Angles
$Z = 180^\circ +^- X$	CLA
$Y + Z = 180^\circ$	Supplementary Angles
$Z = 180^\circ +^- Y^\circ$	CLA
$180^\circ +^- X = 180^\circ +^- Y$	TP
$^-X =^- Y$	CLA
$X = Y$	CLM

The following examples will help appreciate this excellent theorem.

(1) Solve for x and find each of the angles

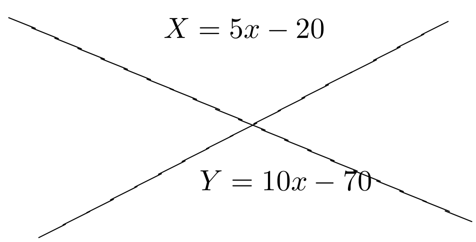


Solution:

$A \cong B$	given
$2x + 30 = 3x - 10$	VAT
$40 = x$	CLA

Therefore, we can substitute x into each of the angles to obtain that each is 120°

(2) Solve for x and find each of the angles

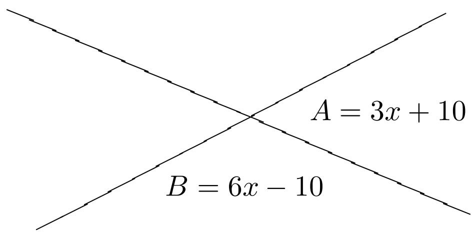


Solution:

$X \cong Y$	given
$5x - 20 = 10x - 70$	VAT
$50 = 5x$	CLA
$10 = x$	CLM

We substitute this in for x to obtain each of the angles is 30°

(3) Solve for x



Solution:

In this case, the given angles are not vertical, they are *supplementary.* By definition of supplementary angles their measurements add up to 180°. Then;

$A\&B$ are supplementary	given
$(3x + 10) + (6x - 10) = 180^\circ$	Def Supp
$9x = 180^\circ$	BI
$x = 20^\circ$	CLM

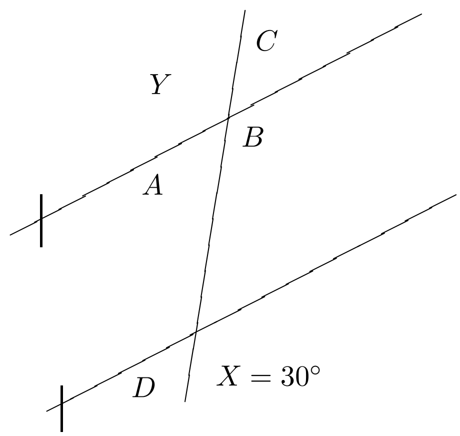
Parallel Lines Theorem [PAL]

According to Euclid's famous masterpiece *Elements-Book I,* definition 23, parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.
The *Parallel Line Theorem [PAL]* says that given two lines and a transversal, either all or none of the following are true.

(1) Alternating Interior Angles are congruent [PAL-ALTI]
(2) Corresponding Angles are congruent [PAL-Cor]
(3) Alternating Exterior Angles are congruent [PAL-ALTE]
(4) The two lines are parallel [PAL-PAR]

We will start with a couple of examples to see how we can use this theorem.

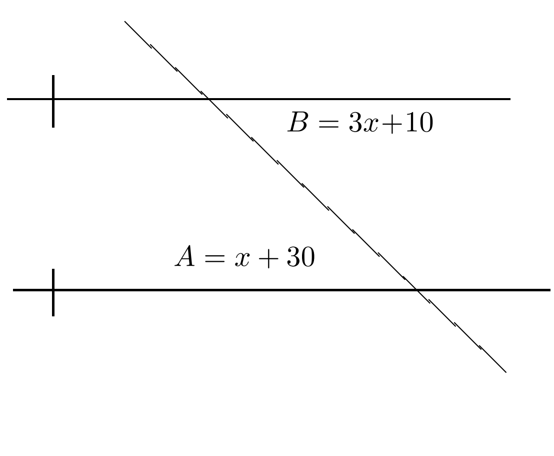
(1) Suppose the following lines are parallel. Solve for all angles.



Solution:

$X = 30°$	given
$B = 30°$	PAL-Cor with X
$Y = 30°$	VAT with B
$C = 150°$	Supplementary with Y
$D = 150°$	PAL-ALTE with C

(2) Solve for x and the angles A and B.



Solution:

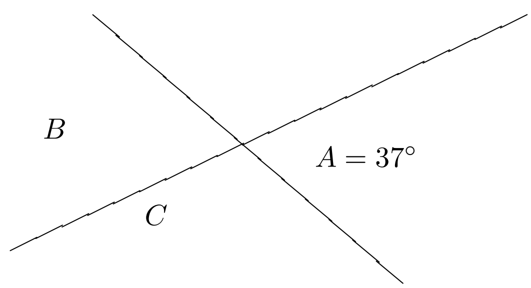
$A \cong B$	PAL-ALTI
$x + 30 = 3x + 10$	Sub
$20 = 2x$	CLA
$10 = x$	CLM

Then we substitute $x = 10$ on each of the angles to obtain that each measures $40°$.

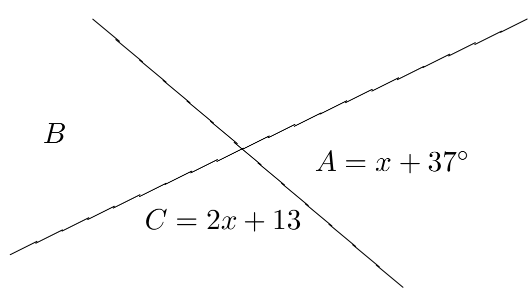
EXCERCISES 9.2

For these exercises, assume the lines are parallel, and all angles are measured in degrees.

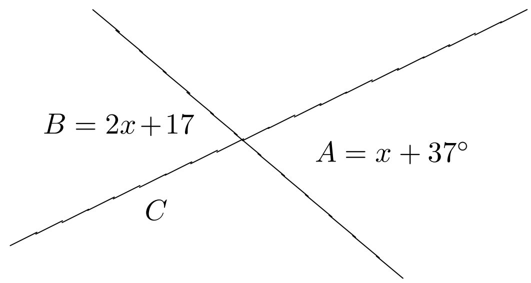
(1) Solve for each angle.



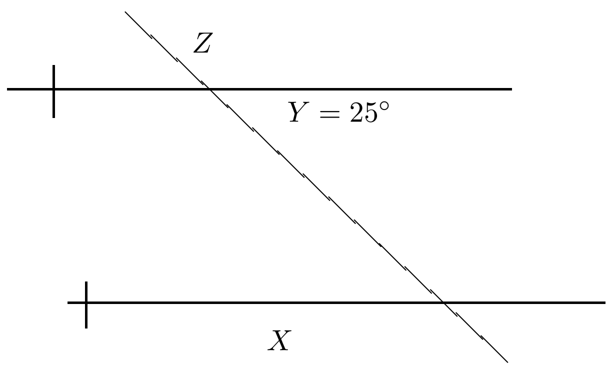
(2) Solve for x, then solve for each angle.



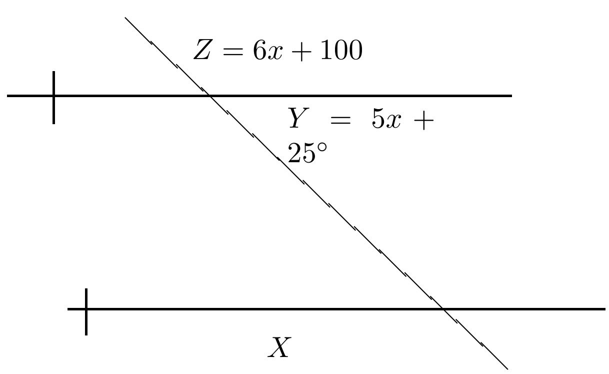
(3) Solve for x, then solve for each angle.



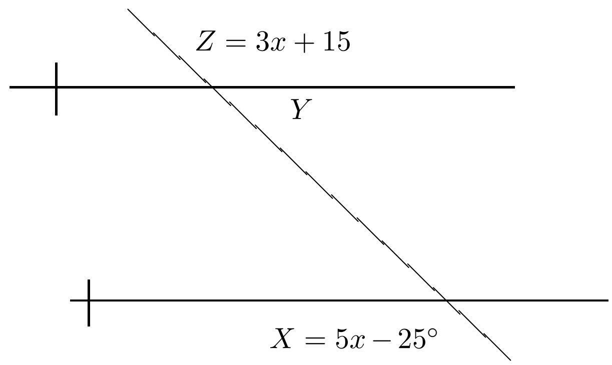
(4) Solve for x, then solve for each angle.



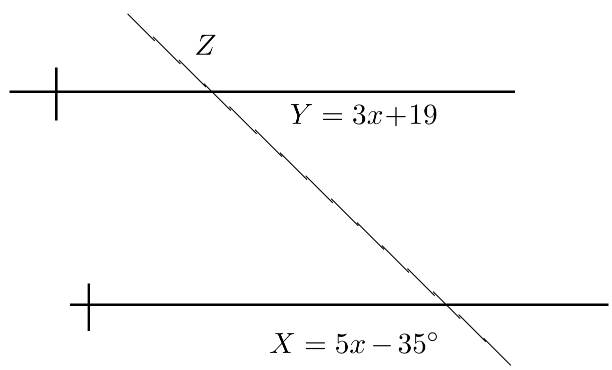
(5) Solve for x, then solve for each angle.



(6) Solve for x, then solve for each angle.



(7) Solve for x, then solve for each angle.



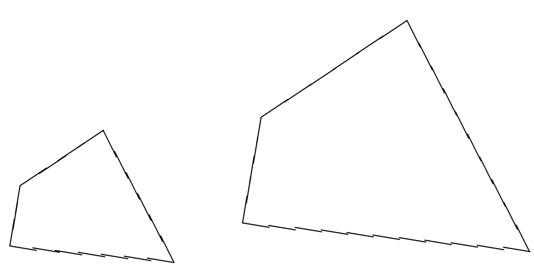
9.3. Similar Shapes

"until one day... nothing happened"

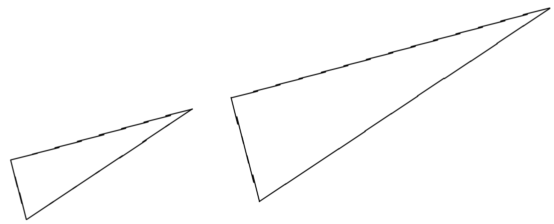
Gameplan 9.3
(1) *What is Similar*
(2) *Similar Triangles*
(3) *Similar Polygons*

What is Similar

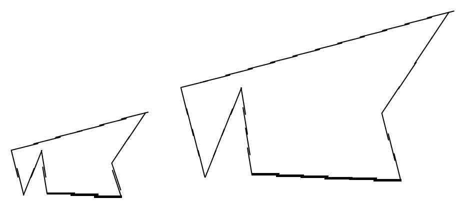
The key feature that makes two shapes *similar* is that their respective lengths are proportional in size. It is important to note that *similar* shapes means everything about the shapes is the same with the only possible variant being size.
The following are examples of similar quadrilaterals:



The following are examples of similar triangles:

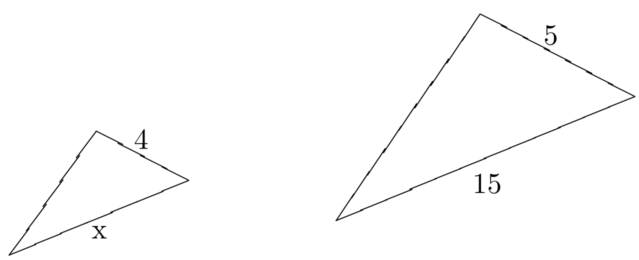


The following are examples of similar polygons:



EXAMPLES: SIMILAR TRIANGLES

(1) Solve for x, you may assume the triangles are similar.



Solution:

To solve for x we will need to exploit the definition of 'similar'. Similar means their sizes are proportional. In words, this means "x is to 15 as 4 is to 5." We turn this into an algebraic equation to begin the solving

$$\frac{x}{15} = \frac{4}{5} \qquad \text{Def Similar}$$

$$x = 15 \cdot \frac{4}{5} \qquad \text{CLM}$$

$$x = 12 \qquad \text{BI}$$

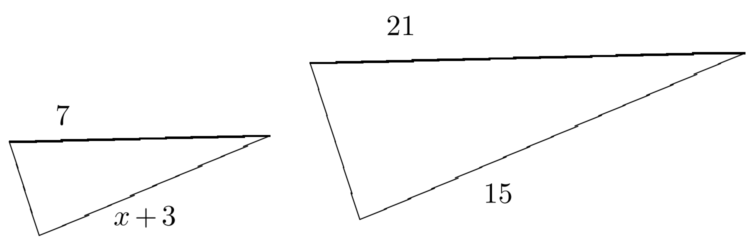
Solution:

Here is an alternative solution. Since the triangles are given to be similar, we can also say "x is to 4 as 15 is to 5" this translates into

$$\frac{x}{4} = \frac{15}{5} \qquad \text{Def Similar}$$
$$x = 4 \cdot 3 \qquad \text{CLM, BI}$$
$$x = 12 \qquad \text{BI}$$

(An expected result!!)

(2) Solve for x, you may assume the triangles are similar.



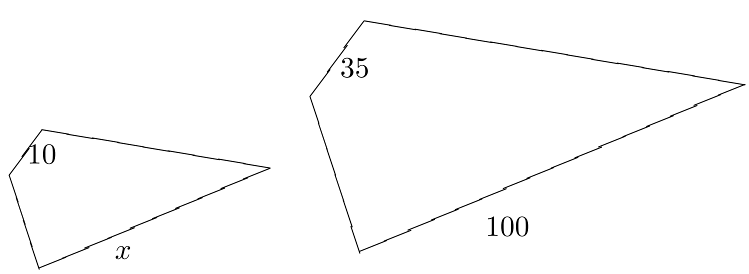
Solution:

To solve for x we will need to exploit the definition of 'similar'. Similar means their sizes are proportional. In words, this means "$x+3$ is to 7 as 15 is to 21." We turn this into an algebraic equation to begin the solving

$$\frac{x+3}{7} = \frac{15}{21} \qquad \text{Def Similar}$$
$$x + 3 = 7 \cdot \frac{15}{21} \qquad \text{CLM}$$
$$x + 3 = 5 \qquad \text{BI}$$
$$x = 2 \qquad \text{CLA}$$

Similar Polygons

(1) Solve for x, you may assume the Polygons are similar.



Solution:

To solve for x we will need to exploit the definition of 'similar'. Similar means their sizes are proportional. In words, this means "x is to 10 as 100 is to 35." We turn this into an algebraic equation to begin the solving

$$\frac{x}{10} = \frac{100}{35} \qquad \text{Def Similar}$$

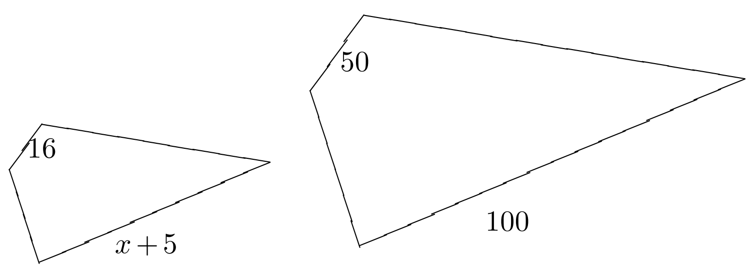
$$x = 10 \cdot \frac{100}{35} \qquad \text{CLM}$$

$$x = \frac{1000}{35} \qquad \text{BI}$$

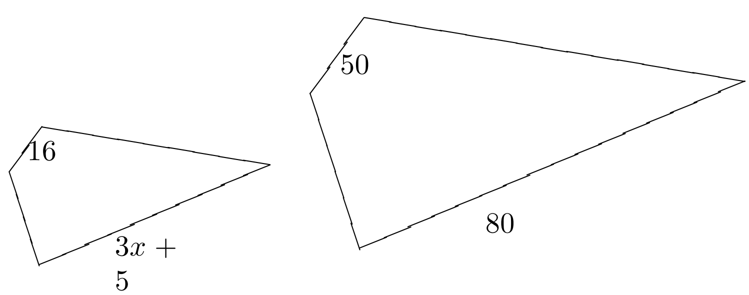
$$x \approx 28.6 \qquad \text{CALC}$$

Excercises 9.3

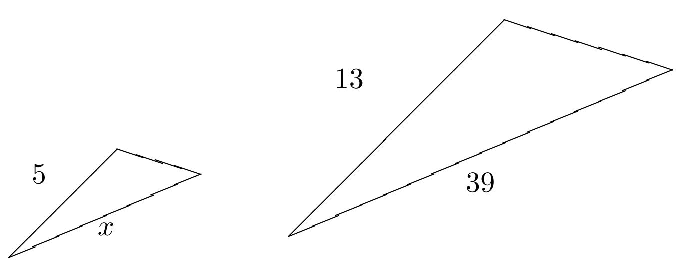
(1) Solve for x, you may assume the Polygons are similar.



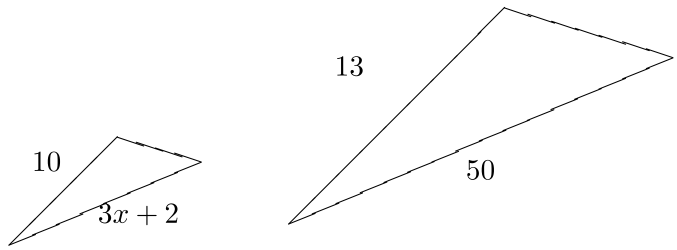
(2) Solve for x, you may assume the Polygons are similar.



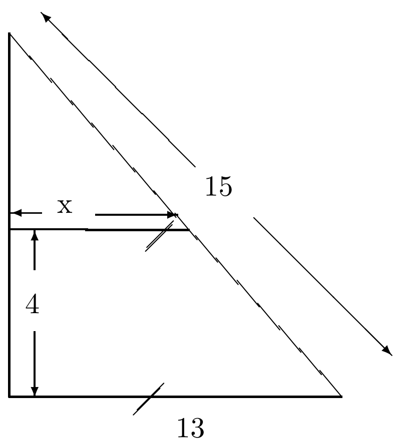
(3) Solve for x, you may assume the Polygons are similar.



(4) Solve for x, you may assume the Polygons are similar.



(5) Solve for x, (yes, I know we did not do one like this in class. Try and see if you can do it by yourself, it will feel very good if you get it. If you don't get it (after a few hours of honest effort), don't worry there is more to life than math and triangles, the answer is useless. It is the *thinking* that is useful.)



9.4. Famous Triangles

"until one day... nothing happened"

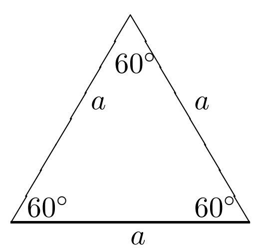
Gameplan 9.4
(1) *Interior Angle Theorem*
(2) *Famous Triangles*
(3) *Pythagoras*
(4) *3060 Triangles*
(5) *45-45 Triangles*

Interior Angles Theorem

Simply states that the sum of the angles inside any (of our) triangle is $180°$. The student is encouraged to derive a convincing argument as to why this is true.

Equilateral Triangles

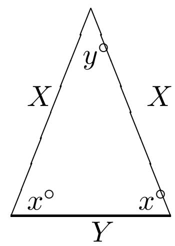
Equilateral triangles are triangles with all sides being equal. It should be noted here that in a triangle there are 3 sides and 3 angles. Each side is associated to one of these angles, the angle opposite from the side. Moreover, large angles produce large (opposite) sides. Thus in an Equilateral triangle, the fact that all sides are equal tells us that all angles are equal as well. Moreover, since all angles are equal, say x degrees, and their sum is $180°$, we must have that $x + x + x = 180$. Thus $3x = 180$, thus $x = 60°$. So we can conclude that for any Equilateral triangle, all interior angles are $60°$.
Typical Equilateral Triangle:



Isosceles Triangles

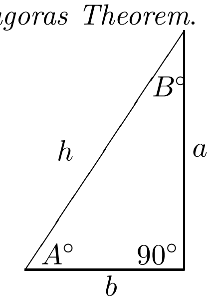
Isosceles Triangles are ones where two of the interior angles are congruent. Note that consequently, two of the sides are also congruent.

Typical Isosceles Triangle:



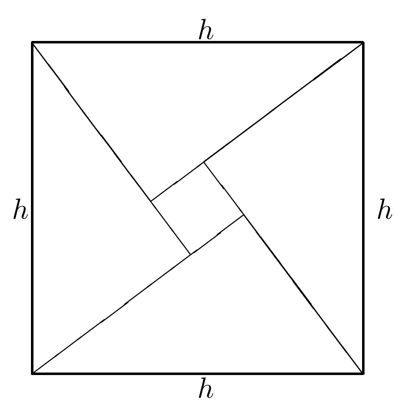
Right Triangles

Perhaps the most famous of all triangles in the world are *Right Triangles*. A right triangle is any triangle that has as one of its interior angles a *right angle*. Said another way, one of its angles is 90°. The longest side is the side opposite 90°. This side is called the *hypotenuse*. The other sides are usually called the short sides or side a and side b. The most important right triangle feature is that, the sides of a right triangle have a very special relationship. This relationship is described by the world famous *Pythagoras Theorem*.
Typical Right Triangle:



Pythagoras

Here is Pythagoras' Idea. Take a right triangle and lay it down on its hypotenuse and take 3 other copies of the triangle and just play around with them. Specifically, he lay them in the following shape.



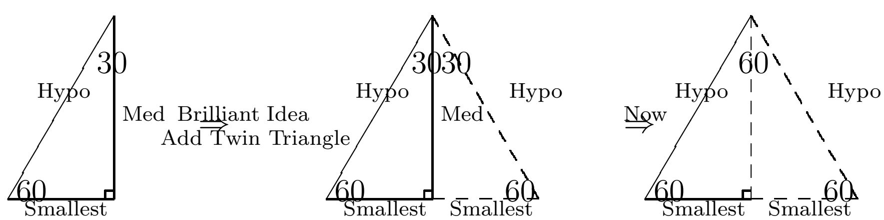
Amazing consequences follow. The Pythagoras's brilliant idea is to calculate the area of this square it a couple of different ways. On the one hand, the are is length times width or $h \cdot h = h^2$. On the other hand, the area is the area the sum of the areas of all 4 triangles plus the area of the little square in the middle. Thus we have:

$$\begin{aligned} h^2 &= \text{Area of Big Square} && A = lw \\ \text{Area of Big Square} &= 4(\text{area of each triangle}) + \text{area of small square} && \text{B.I.} \\ &= 4(\frac{a \cdot b}{2}) + (b-a)^2 && \text{B.I.} \\ &= 2ab + b^2 - 2ab + a^2 && \text{FOIL} \\ &= b^2 + a^2 && \text{B.I.} \\ h^2 &= b^2 + a^2 && \text{transitivity} \end{aligned}$$

An excepted result!! We have just proven the world famous Pythagoras Theorem (this is a BIG theorem).It says *the square of the hypotenuse of a right triangle is the sum of the squares of each of the smaller sides.*

30-60 Triangles

30-60 triangles also enjoy a very place in every scientist's heart. First, they are right triangles, since one angle is 30, another 60, the last angle must e 90 degrees. Thus, they enjoy all the benefits of a right triangle, i.e. Pythagoras Theorem. In addition, their sides have another very special relationship. We will explore this relationship by adding a twin triangle next to it. Observe:
Typical 30-60 Triangle:



And now.... we let the figuring begin!!!! How does the small side compare to the hypotenuse? The last triangle is an Equilateral triangle since all its angles are equal. Therefore, all sides are equal. Therefore, we get the very special relationship for 30-60 triangles:

The small side is half the hypotenuse

But we are not done. For ease of writing we will call the sides h for the hypotheses, m for the medium side *which is always the one across from the* $60°$, and we will use s for the smallest side which is always the one across the smallest angle, $30°$. Now, we resume to the figuring. Concentrating on the original triangle we use our new-found knowledge that $h = 2s$ (hypotenuse is twice the small side). We will call this excellent theorem the 3060-Small Theorem [3060S].

$$h = 2s \qquad [3060S]$$

What about the medium side? We apply the powerful Pythagoras Theorem to conclude...

$h^2 = s^2 + m^2$	P.T.
$(2s)^2 = s^2 + m^2$	Substitute
$(2s)(2s) = s^2 + m^2$	+Expo
$4s^2 = s^2 + m^2$	+Expo
$3s^2 = m^2$	C.L.A
$m = \pm\sqrt{3s^2}$	SRP
$m = \sqrt{3s^2}$	m is a Positive length
$m = \sqrt{s^2}\sqrt{3}$	RP=PR
$m = s\sqrt{3}$	RAD

Thus we get another very important relationship. Namely,

The medium side is $\sqrt{3}$ times the smallest side

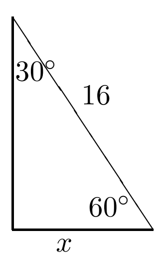
In math symbols,

$$m = s\sqrt{3} \qquad [3060M]$$

These relationships amongst the sides of a right triangle are very useful in that once we know *any* one of the sides, we can easily figure out the rest of the sides. That is it! all it takes is one side, and the rest is Duck Soup!!!

EXAMPLES

(1) Solve for x



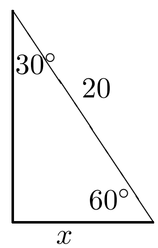
Solution:

We first identify which side is which. Since '16' is opposite the right angle, it must be the hypothenuse. Since x is opposite the $30°$ angle, it must be the small side. Now we use the 3060S theorem to begin.

$h = 2s$	3060S
$16 = 2x$	Sub
$8 = x$	CLM

Done!

(2) Solve for x



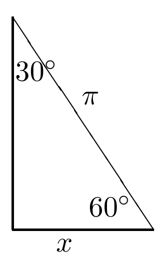
Solution:

We first identify which side is which. Since '20' is opposite the right angle, it must be the hypothenuse. Since x is opposite the 30° angle, it must be the small side. Now we use the 3060S theorem to begin.

$h = 2s$	3060S
$20 = 2x$	Sub
$10 = x$	CLM

Done!

(3) Solve for x



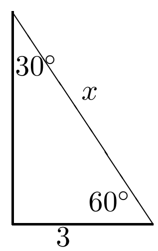
Solution:

We first identify which side is which. Since 'π' is opposite the right angle, it must be the hypothenuse. Since x is opposite the 30° angle, it must be the small side. Now we use the 3060S theorem to begin.

$$h = 2s \quad \text{3060S}$$
$$\pi = 2x \quad \text{Sub}$$
$$\frac{\pi}{2} = x \quad \text{CLM}$$

Done!

(4) Solve for x



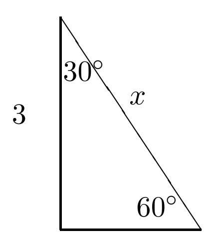
Solution:

We first identify which side is which. Since 'x' is opposite the right angle, it must be the hypothenuse. Since 3 is opposite the 30° angle, it must be the small side. Now we use the 3060S theorem to begin.

$$h = 2s \quad \text{3060S}$$
$$x = 2 \cdot 3 \quad \text{Sub}$$
$$x = 6 \quad \text{BI}$$

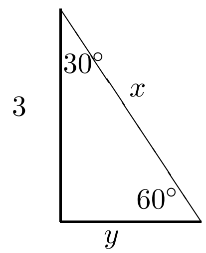
Done!

(5) Solve for x



Solution:

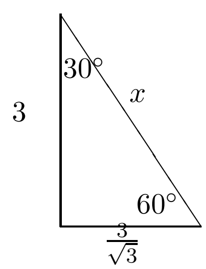
We first identify which side is which. Since 'x' is opposite the right angle, it must be the hypothenuse. Since 3 is opposite the 60° angle, it must be the medium side. We don't have a theorem which compares the medium side to the hypothenuse. Nevertheless, we know how the small side compares to the hypothenuse and how the small side compares to the medium, thus we will first find the small side. We will first label it y to obtain



We now find the small side.

$$\begin{aligned} m &= s\sqrt{3} && \text{3060M} \\ 3 &= s\sqrt{3} && \text{Sub} \\ x &= \frac{3}{\sqrt{3}} && \text{CLM} \end{aligned}$$

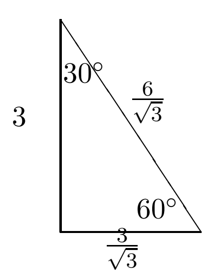
We now update our triangle with the resolved small side..



And now proceed to use use the small side to find the hypothenuse.

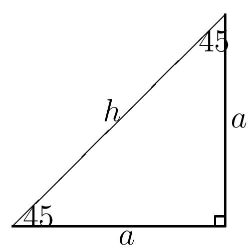
$$\begin{aligned} h &= 2s && \text{3060S} \\ x &= 2 \cdot \left(\frac{3}{\sqrt{3}}\right) && \text{Sub} \\ x &= \frac{6}{\sqrt{3}} && \text{BI} \end{aligned}$$

As a final solution we offer



45-45 Triangles

Ultimately, we get the same conclusion for the 45-45 triangles. That is, given any one of the sides we can immediately determine the other two sides. However, the actual figuring out of the sides will be nowhere near as exiting as the case was for 30-60 triangles. For starters, the 45-45 triangles are isosceles. Two of the angles are equal thus the two corresponding sides are equal. So we have:
Typical 45-45 Triangle:



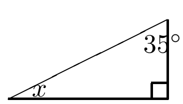
We invoke Pythagoras....

$$\begin{aligned} h^2 &= a^2 + a^2 && \text{P.T.} \\ h^2 &= 2a^2 && \text{B.I.} \\ h &= \pm\sqrt{2a^2} && \text{SRP} \\ h &= \sqrt{2a^2} && h \text{ is positive} \\ h &= a\sqrt{2} && \text{B.I.} \end{aligned}$$

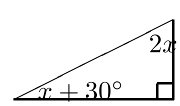
This tells us all we need to know. The two side are equal and to get the hypotenuse we just multiply by $\sqrt{2}$. Again, with any one of the sides we can determine the rest of the sides. We will call this the 4545 Theorem [4545T].

Excercises 9.4

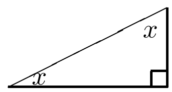
(1) Solve for x on the following right triangles.



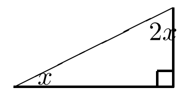
(a)



(b)

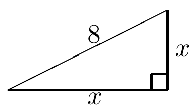


(c)

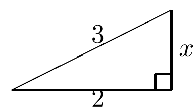


(d)

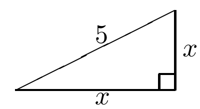
(2) Solve for x on the following right triangles.



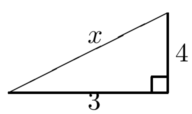
(a)



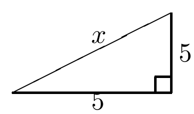
(c)



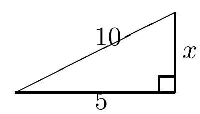
(e)



(b)



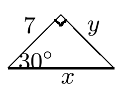
(d)



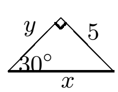
(f)

(3) Figure out all sides of the triangles.

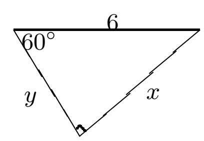
(a)



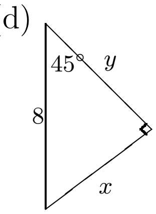
(b)



(c)

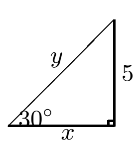


(d)

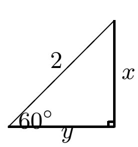


(4) Figure out all sides of the triangles.

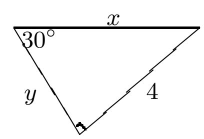
(a)



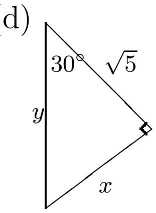
(b)



(c)



(d)



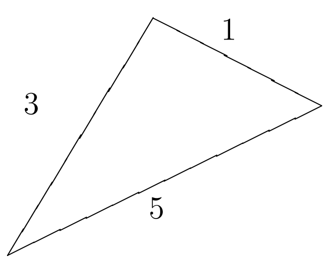
9.5. Perimeter

"until one day... nothing happened"

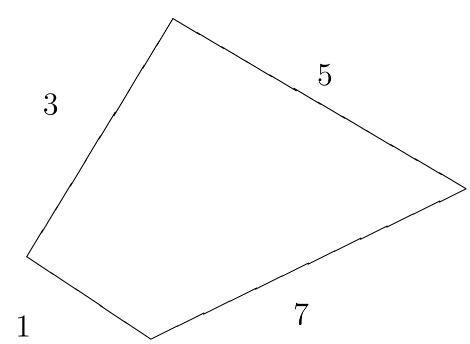
Gameplan 9.5
(1) *Perimeter: Polygons*
(2) *Circumference*
(3) *Arc-length*

Perimeter: Polygons

A three-sided figure is generally called a triangle. A four-sided figure is called a quadrilateral, five-sided shapes are called pentagons, then hexagons, etc. All of these belong to the family of polygons. Given any polygon, we will define its perimeter to be the sum of the lengths of each of its sides.



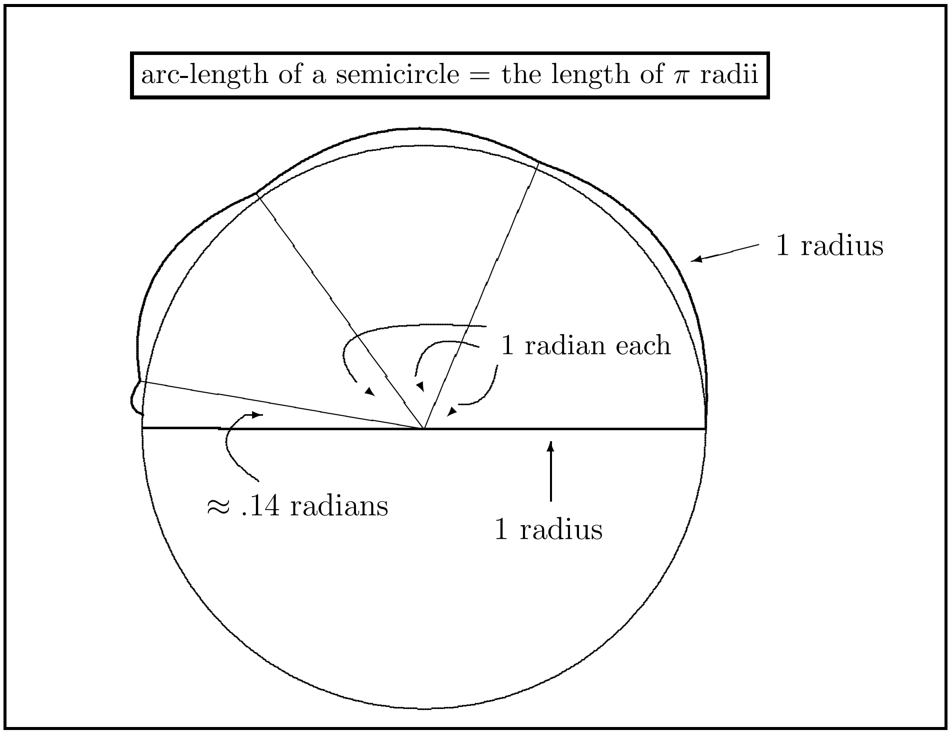
Perimeter = 3+1+5 = 9



Perimeter = $1 + 3 + 5 + 7 = 16$

Circumference

Circumference is the circle's version of perimeter. It is the amount of length required to go around the circle. There are those that will tell you to calculate the circumference of a circle we can use π. It's a bit misleading. Much like saying dirt was invented so we could use brooms. Brooms were invented to sweep dirt much like π was born calculate the circumference of a circle. Indeed, by definition, we can take π to be the ratio between the arc-length of a semicircle and the radius. More generally,



We continue this idea to develop it into a recipe for the circumference of a circle. Namely, if a half-circle has arc-length equal to π times the length of the radius, then the entire circle would have twice as much arc-length. This means

if

$$\boxed{\text{arc-length of a semicircle} = \text{the length of } \pi \text{ radii}}$$

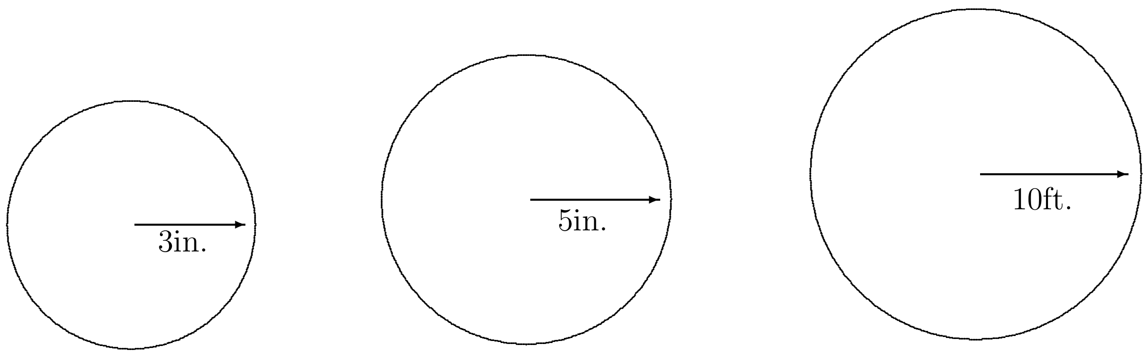
then

$$\boxed{\text{arc-length of a circle} = \text{twice the length of } \pi \text{ radii}}$$

For convenience we will refer to the circumference of a circle as c while we will the length of the radius will always be called r for short. Thus the above statement can be re-written as the world know, super famous circumference function for a circle. We will call this the circumference theorem [CirT]

$$\boxed{C = 2\pi r} \qquad \text{[CirT]}$$

EXAMPLES



$C = 2\pi \cdot 3in. = 6\pi in$ $C = 2\pi \cdot 5in = 15\pi in$ $C = 2\pi \cdot 10ft = 20\pi ft$

ARC-LENGTH

We will now entertain the idea of finding the length of a proportion of the circles total circumference. The idea here is to use the definition of a radian. Recall *a radian* is the angle created by distance of one radius along the arc-length of a circle. The traditional variable representing arc-length is s while the number of radians is usually represented by the Greek letter *theta*, θ. In word we could represent the relationship between each of these by

# of radians = arc-length divided by # radii that fit along the arc-length

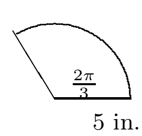
Using variables the above equation reads,

$$\boxed{\theta = s/r} \qquad \text{[Def of Radians]}$$

We can then solve for s (arc-length) and we will have one of the most famous formulas ever. The formula for the arc-length. We shall call it the Arc-Length Theorem [ArcL].

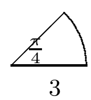
$$\boxed{s = r\theta} \qquad \text{[ArcL]}$$

EXAMPLES

(1) Find the arc-length for 

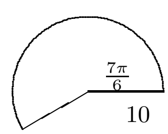
Solution:

$$\begin{aligned} s &= r\theta && \text{ArcL} \\ &= 5in.\left(\frac{2\pi}{3}\right) && \text{Sub} \\ &= \frac{10\pi}{3}in. && \text{BI} \\ &\approx 10.5in && \text{calc} \end{aligned}$$

(2) Find the arc-length for 

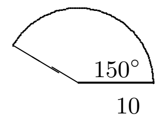
Solution:

$$\begin{aligned} s &= r\theta && \text{ArcL} \\ &= 3\left(\frac{\pi}{4}\right) && \text{Sub} \\ &= \frac{3\pi}{4} && \text{BI} \\ &\approx 2.4 && \text{calc} \end{aligned}$$

(3) Find the arc-length for 

Solution:

$$\begin{aligned} s &= r\theta && \text{ArcL} \\ &= 10\left(\frac{7\pi}{6}\right) && \text{Sub} \\ &= \frac{70\pi}{6} && \text{BI} \\ &\approx 36.7 && \text{calc} \end{aligned}$$

(4) Find the arc-length for 

Solution:

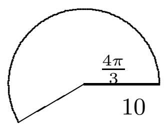
Note [ArcL] was designed and formulated to be used with radians. It will not give the correct arc-length if the angle θ is expressed in degrees. However, we can always convert the degrees to radians and then use [ArcL]. Recall 150° converts to $\frac{5\pi}{6}$ radians. So we can set $\theta = \frac{5\pi}{6}$ and proceed:

$$\begin{aligned} s &= r\theta && \text{ArcL} \\ &= 10\left(\frac{5\pi}{6}\right) && \text{Sub} \\ &= \frac{50\pi}{6} && \text{BI} \\ &\approx 26.2 && \text{calc} \end{aligned}$$

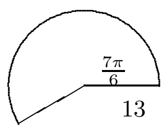
Excercises 9.5

Find perimeter on polygons and the arc-length on on circles. (drawings not to scale)

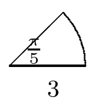
(1)



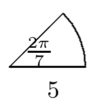
(2)



(3)



(4)



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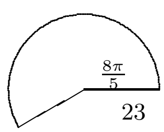
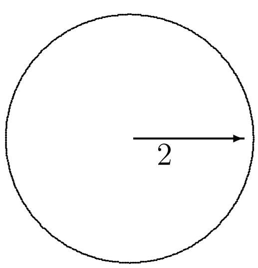
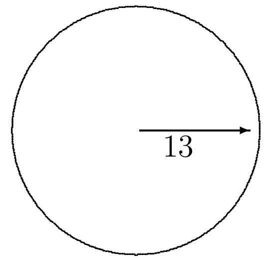


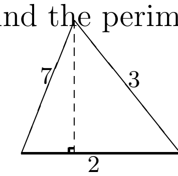


(6)

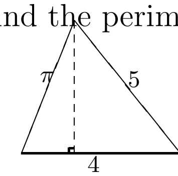


(7)

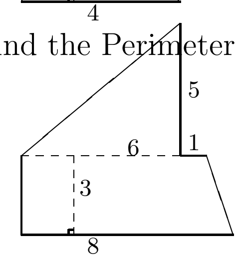
(8) Find the perimeter



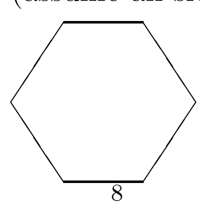
(9) Find the perimeter



(10) Find the Perimeter



(11) Find the Perimeter
(assume all sides/anlges equal)



9.6. Area

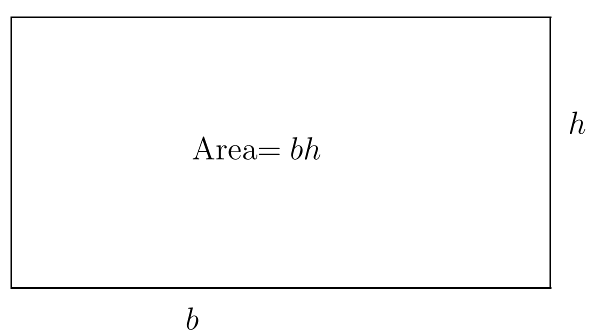
"until one day... nothing happened"

Gameplan 9.6

(1) *Area: Rectangles*
(2) *Area: Triangles*
(3) *Area: Trapezoids*
(4) *Area: Circles*
(5) *Area: Sectors*

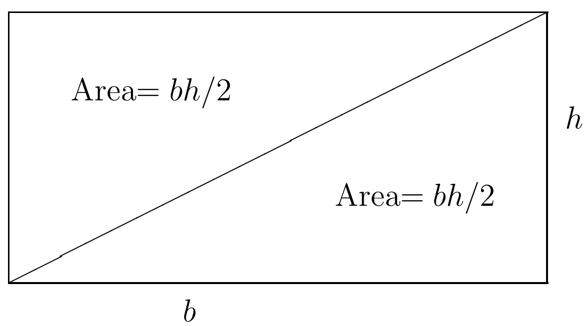
AREA: RECTANGLE

We define area first for a rectangle as the product of its width times its length.

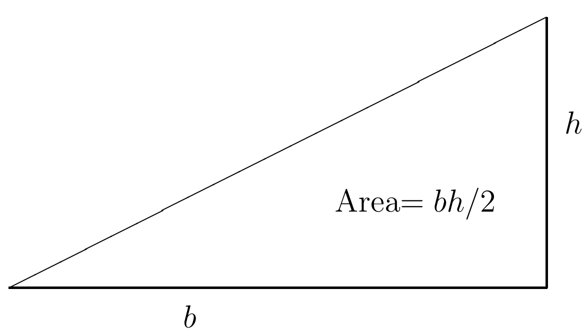


AREA: TRIANGLE

To obtain the area of triangle, we note triangles are just half-quadrilaterals. So we take half of the area of the corresponding quadrilateral to obtain the famous formula for the area of a triangle, area = base times hight divided by 2, $A = \frac{bh}{2}$.

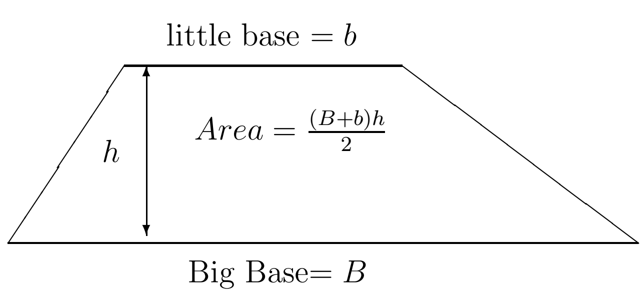


The recipe works to find the area of any triangle. Always the base time the hight divided by two.



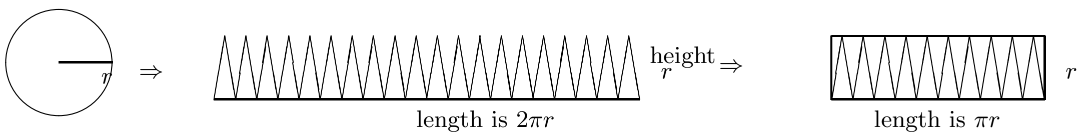
AREA: TRAPEZOIDS

A trapezoid is a 4 sided shape where two of the sided are parallel. The area is given by taking the average of the two bases times the hight. The proof of why this is a sensible formula for the area is left for the reader as an important exercise.



Area: Circles

To determine the area of a circle we may consider the following idea.

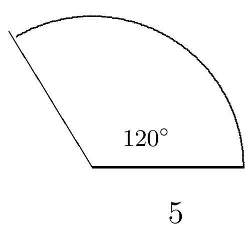


These pictures suggest that the area of the circle can be represented in any of the three above ways of which the last is the easiest to calculate since it is a rectangle of height r and length πr. In fact, this is the true are of a circle. A formal proof will have to wait for a more advanced course, for now we will take faith in the above argument and conclude the world famous *Area of a Circle Theorem (ACT)*:

$$'\text{Area of Circle with radius } r' = \pi r^2$$

Area: Sectors

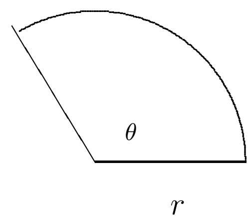
Sectors are just portions of circles. We will first find the area of one particular sector. We will then generalize the idea to cook up an excellent recipe to find the area of any sector with the greatest of ease.
Consider find the area of the following sector.



Note the sector has radius of 5 units and measures 120 degrees. Another way to look at it is to see it: The entire circle measures 360 degrees. This piece of it is just 120 degrees. Then this sector is exactly one third of the entire circle. Here is the strategy. We will find the area of the entire circle, then we will just take one third of of the entire area, and we will have the exact area of the sector.
First we find the area of an entire circle with the same radius, namely,
$A = \pi(5)^2 = 25\pi \approx 78.5$. We then take one third of that, and this will give us the area of our sector,

$$\text{area of sector } = 78.5 \div 3 = 26.18$$

Now for the general recipe, given a sector that measures θ radians, and radius r, will find the area.



The entire area of such a circle would be $A = \pi r^2$. This particular portion is just θ radians out of 2π total radians. Thus the proportion is given by the fraction $\frac{\theta}{2\pi}$. Then the corresponding proportion of the area would be given by "sector area = (entire area)(proportion)" which translates into the world famous sector area formula:

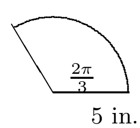
"sector area = (entire area)(proportion)"

$$\boxed{SArea = \pi r^2 \cdot \left(\tfrac{\theta}{2\pi}\right)}$$

Which can by simplified to

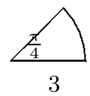
$$\boxed{SArea = r^2 \left(\tfrac{\theta}{2}\right)}$$

Examples

(1) Find the Area for Sector 

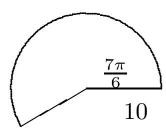
Solution:

$$\begin{aligned} Sarea &= r^2 \left(\frac{\theta}{2}\right) && \text{SArea} \\ &= (5in.)^2 \left(\frac{2\pi}{6}\right) && \text{Sub} \\ &= \frac{50\pi}{6} in^2 && \text{BI} \\ &\approx 26.18 in^2 && \text{calc} \end{aligned}$$

(2) Find the Area for Sector 

Solution:

$$\begin{aligned} Sarea &= r^2\left(\frac{\theta}{2}\right) && \text{SArea}\\ &= (3in.)^2\left(\frac{\pi}{8}\right) && \text{Sub}\\ &= \frac{9\pi}{8}in^2 && \text{BI}\\ &\approx 3.53in^2 && \text{calc} \end{aligned}$$

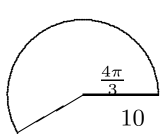
(3) Find the Area for Sector 

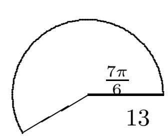
Solution:

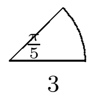
$$\begin{aligned} Sarea &= r^2\left(\frac{\theta}{2}\right) && \text{SArea}\\ &= (10in.)^2\left(\frac{7\pi}{12}\right) && \text{Sub}\\ &= \frac{700\pi}{12}in^2 && \text{BI}\\ &\approx 183.26in^2 && \text{calc} \end{aligned}$$

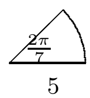
EXCERCISES 9.6

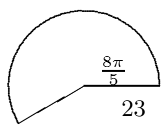
Find the Area

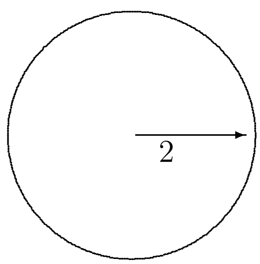
(1) 

(2) 

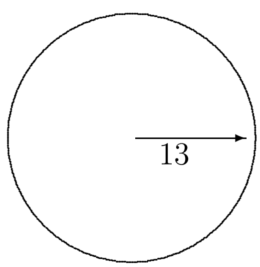
(3) 

(4) 

(5) 

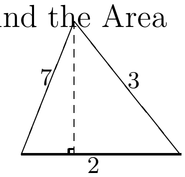


(6)

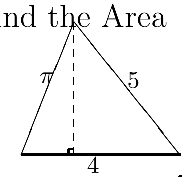


(7)

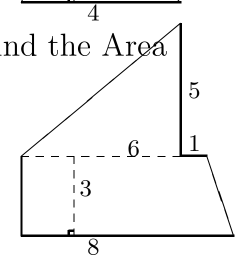
(8) Find the Area



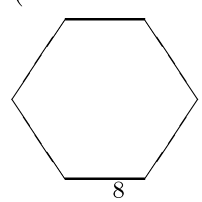
(9) Find the Area



(10) Find the Area



(11) Find the Area
(assume all sides/anlges equal)



(12) Which is a better deal for \$9.99: 2 medium pizzas (10 in. diameter) or 1 large (15 in. diameter)?

CHAPTER 10

Functions

10.1. Intro

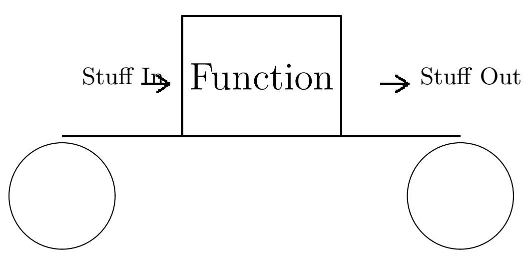
"until one day... nothing happened"

Gameplan 10.1
(1) *Function*
(2) *Famous Functions*
(3) *Evaluate*
(4) *Arithmetic Of Functions*

FUNCTION

The concept of a *function* is essential in the study of mathematics. Here, we will study some of the more common ways of describing a function.

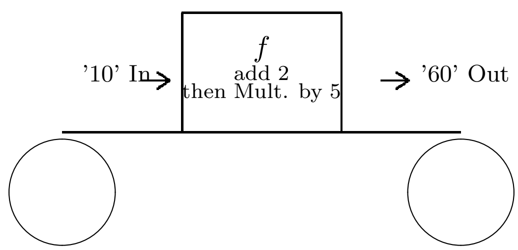
A FUNCTION AS A MACHINE



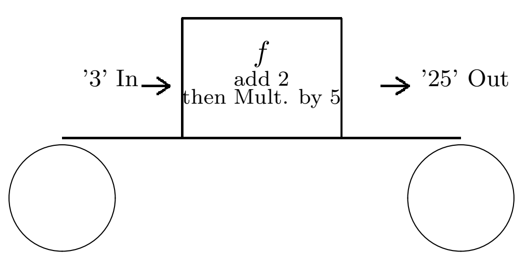
This is an excellent way of thinking about functions. Think about a function as three pieces of information, the stuff we put in, the machine it goes into, and the stuff that comes out. For example, we can take some stuff like numbers, put them through a machine that may multiply, subtract or in some other way manipulate then finally produce an output. Of course, the stuff does not have to be numbers. You could have a function where you feed *water in* and get *ice cubes out*. There are endless possibilities for the *stuff in*, for the *machine* as well as for the *stuff out,* yet all functions must contain these three key ingredients.

A FUNCTION AS A RECIPE

Another excellent way one can think about functions is to think about them as recipes. They contain three crucial pieces of information: The *ingredients that you put in*, *the procedure* that tells you what to do with the ingredients, and *the end product*. All these terms have more professional names. For examples, the stuff that you put in is usually called the *Domain* of the function. The rule, or procedure is usually called the *Unambiguous Rule*. The output, the stuff you get out of it is usually called the *Range* or the *Co-Domain*. To describe a function completely, one must describe these three pieces of information. The domain, range, and the unambiguous rule.
For Example: Suppose we define the domain to be the Real Numbers, the Range to be also the Real number and suppose we define the recipe to be "to each real number x, add 2, then multiply by 5". Because all three pieces of information are given, this is a perfect example of a function, we'll call this function f. Using our picture, let us see what this function does to a number like 10:



Let us see what f does to 3:

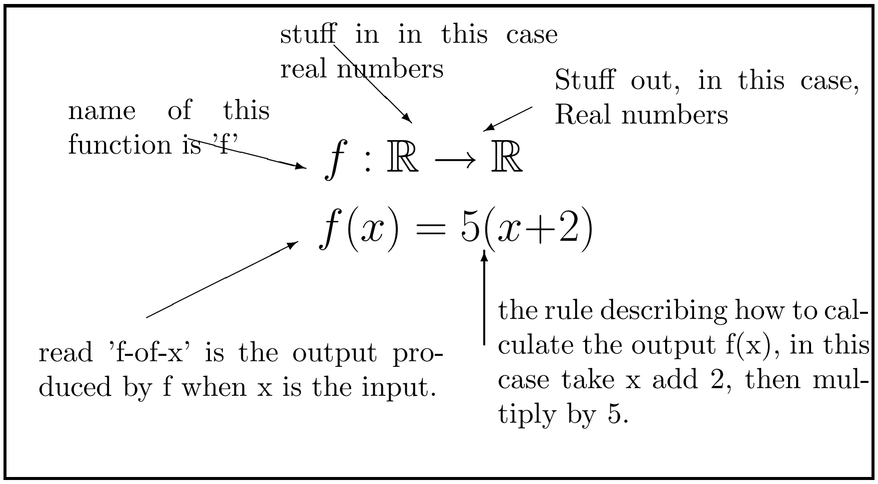


Of course, this way of describing functions has it's drawbacks. If you are not a good artist or in the mood for drawing machines all day, you may appreciate the great developments made to express functions in a clear an concise way. Over the years the following conventions have been adopted.

The function f takes any Real Number Input x, adds 2 to it, then multiplies by 5, to get a Real Number output.

Is equivalent to saying:

The anatomy of a function



The first line tells you, the name of this function is 'f', the stuff we will put 'in' is Real Numbers, the stuff we will get 'out' is Real Numbers. The second line describes the recipe, add 2 then multiply by 5. Note '$f(x)$' DOES NOT mean 'f times x'. It means "what f does to x", and it is read 'f-of-x'. The above function machine picture is replace by the equations that describe what f does to numbers. Note x is used here a place holder. We could have easily described the same recepie using t to describe what f does, namely, $f(t) = 5(t + 2)$. In fact we could have used the famous 'blah' to describe what f does to any number 'blah', namely

$$f(blah) = 5(blah + 2)$$

A couple numerical examples are in order.

$$f(10) = 5(10 + 2) = 5(12) = 60$$

and

$$f(3) = 5(3 + 2) = 5(5) = 25$$

while

$$f(\heartsuit) = 5(\heartsuit + 2)$$

Also note that up to this point most of the algebraic systems we have studied were commutative. That is, we freely assumed the Commutativity Law of Addition and the Commutativity Law of Multiplication. In other words, if x is any *number* then

$$3x = x3 \qquad [CoLM]$$

Please make a note of it: *Functions are not numbers!* They are recipes with three pieces of information, They are not numbers, they are not numbers, they are not numbers, do not expect them to follow the real number axioms. They are generally not commutative nor do they follow the distributive law. For starters, if f is a function, we will always write 'f-of-x' as $f(x)$ and never as $(x)f$. Moreover consider the function

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = x^2$$

If we will calculate $f(x)$, then $f(y)$ and $f(x+y)$ leading us to a monumental observation. First note $f(x) = x^2$ while $f(y) = y^2$ and $f(x+y) = (x+y)^2 = x^2 + 2xy + y^2$ Then...

$$f(x+y) = x^2 + 2xy + y^2$$

while

$$f(x) + f(y) = x^2 + y^2$$

and the monumental observation is that

$$f(x+y) \neq f(x) + f(y)$$

I told you! function do not admit the Distributive Law, they do not behave like numbers. If we had some number instead of f we would have no problem using the distributive law...

$$3(x+y) = 3x + 3y \qquad [(noproblem)\, DL]$$

Having said that, we turn out attention to the near future. There are a couple of things that any self-respecting algebra student needs to learn about functions. One is to become familiar with at least the very most *famous functions.* Second, to become very good at *evaluation functions.* And third, to become familiar with the most famous binary operation in the world of functions, *composition.* The rest of this section is devoted to these ideas.

Famous Functions

Type of Function	Example	Typical Domain	Typical Range	Typical Graph
Linear Functions	$f(x) = 3x + 2$	$\mathbb{R}$	$\mathbb{R}$	
Quadratic Functions	$f(x) = 3x^2 + 2x - 3$	All $\mathbb{R}$	Part of $\mathbb{R}$	
Cubic Functions	$f(x) = x^3 - 2t^2 + 1$	All $\mathbb{R}$	All $\mathbb{R}$	
Rational Functions	$f(x) = \frac{x^2+1}{5x-5}$	Part of $\mathbb{R}$	All $\mathbb{R}$ Except points where denominator is zero	
Exponential Functions	$f(x) = 2^x$	All $\mathbb{R}$	Usually, only $\mathbb{R}^+$	
Logarithmic Functions	$f(x) = \log_2 x$	Only $\mathbb{R}^+$	All $\mathbb{R}$	

EVALUATE

One of the most important skills needed in real life is to be able to *evaluate* functions. In other words, once we specify a particular rule domain and range, we need to be able to apply the rule efficiently and effortlessly to any element in the domain and determine which element in the range it gets mapped to. Consider the following examples.

(1) Suppose $f(x) = x^2 + 3$, evaluate the following:

(a) $f(2)$
(b) $f(3)$
(c) $f(5)$
(d) $f(6)$
(e) $f(t)$
(f) $f(cat)$

Solution:

(a) $f(2) = (2)^2 + 3 = 7$
(b) $f(3) = (3)^2 + 3 = 12$
(c) $f(5) = (5)^2 + 3 = 28$
(d) $f(6) = (6)^2 + 3 = 39$
(e) $f(t) = (t)^2 + 3 = t^2 + 3$
(f) $f(cat) = (cat)^2 + 3$

(2) Suppose $f(x) = x^2 + 3$, evaluate the following:
(a) $f(t+1)$
(b) $f(x+h)$
(c) $f(f(x))$

Solution:

(a) $f(t+1)$

$$\begin{aligned} f(t+1) &= (t+1)^2 + 3 && \text{Def of } f \\ &= t^2 + 2t + 1 + 3 && \text{PP2} \\ &= t^2 + 2t + 4 && \text{PP2} \end{aligned}$$

(b) $f(x+h)$

$$\begin{aligned} f(x+h) &= (x+h)^2 + 3 && \text{Def of } f \\ &= tx^2 + 2xh + h^2 + 3 && \text{PP2} \end{aligned}$$

(c) $f(f(x))$

$$\begin{aligned} f(f(x)) &= (f(x))^2 + 3 && \text{Def of } f \\ &= \left(x^2 + 3\right)^2 + 3 && \text{Sub } f \\ &= x^4 + 2 \cdot 3x^2 + 3^2 + 3 && \text{PP2} \\ &= x^4 + 6x^2 + 12 && \text{PP2} \end{aligned}$$

Arithmetic Of Functions

Adding/Subrtracting Functions

Just as we can add real or complex numbers together we can also impose an addition law for functions. That is, if the domains and ranges are compatible for the corresponding functions we can in fact add the two functions to get a new functions. Let $f(x) = 3x + 1$

and $g(x) = x^2$ with domain and range Real Numbers. Then we define the sum function $f + g : \mathbb{R} \to \mathbb{R}$ with rule $(f+g)(x) = f(x) + g(x)$ Said another way, $(f+g)(x) = 3x + 1 + x^2$. The difference, product and quotient functions are defined in a similar way.

Example

Suppose $f(x) = x^2$ and $g(x) = 2x - 1$ and $h(x) = \frac{1}{x}$ all with domain and range $\mathbb{R}$

(1) find $(f+g)(2)$
(2) find $(f+g)(x)$
(3) find $3f(x)$
(4) find $3f(5)$
(5) find $fh(x)$
(6) find $\frac{f}{g}(5)$
(7) find $\frac{f}{g}(t)$

Solution:

(1) $(f+g)(2) = f(2) + g(2) = (2)^2 + 2(2) - 1 = 7$

(2) $(f+g)(x) = f(x) + g(x) = x^2 + 2x - 1$

(3) $3f(x) = 3x^2$

(4) $3f(5) = 3 \cdot 5^2 = 75$

(5) $fh(x) = f(x)h(x) = x^2 \cdot \frac{1}{x} = \frac{x^2}{x}$

(6) $\frac{f}{g}(5) = \frac{f(5)}{g(5)} = \frac{5^2}{2 \cdot 5 - 1} = \frac{25}{9}$

(7) $\frac{f}{g}(t) = \frac{t^2}{2t-1}$

Composition Defined

If $f : A \to B$ and $g : B \to C$ then we define a new function called the "the composition of f and g" as:

$$g \circ f : A \to C \quad \text{with } g \circ f(x) = g(f(x))$$

Examples

Suppose $f(x) = x^2$ and $g(x) = 2x - 1$ and $h(x) = \frac{1}{x}$. Determine:

(1) $f \circ h(3)$

Solution:

$$\begin{aligned}
f \circ h(3) &= f(h(3)) && \text{def of } f \circ h \\
&= f(\frac{1}{3}) && \text{def of } h \\
&= (\frac{1}{3})^2 && \text{def of } f \\
&= \frac{1}{9} && \text{By Inspection}
\end{aligned}$$

(2) $h \circ f(3)$

Solution:

$$\begin{aligned}
h \circ f(3) &= h(f(3)) && \text{def of } h \circ f \\
&= h(3^2) && \text{def of } f \\
&= h(9) && \text{BI} \\
&= \frac{1}{9} && \text{def of } h
\end{aligned}$$

(3) $f \circ g(5)$

Solution:

$$\begin{aligned}
f \circ g(5) &= f(g(5)) && \text{def of } f \circ g \\
&= f(2 \cdot 5 - 1) && \text{def of } g \\
&= f(9) && \text{BI} \\
&= 81 && \text{def of } f
\end{aligned}$$

(4) $g \circ f(5)$

Solution:

$$\begin{aligned} g\circ f(5) &= g(f(5)) && \text{def of } g\circ f \\ &= g(2(5)-1) && \text{def of } g \\ &= g(9) && \text{BI} \\ &= 2\cdot 9-1 && \text{def of } g \\ &= 17 && \text{def of } g \end{aligned}$$

Note, in general $f\circ g \neq g\circ f$ as evident from the these examples.

EXCERCISES 10.1

Let $h(x)=3x+1$ while $k(x)=x^3+1$, $f(x)=3x$ all with domain and range $\mathbb{R}$. Evaluate

(1) $f(2)$
(2) $h(3)$
(3) $k(5)$
(4) $h(2+t)$
(5) $g\circ f(2)$
(6) $k\circ h(3)$
(7) $f\circ k(5)$
(8) $h\circ h(2+t)$
(9) $f(h(2)$
(10) $h(f(3))$
(11) $k(5)+h(1)$
(12) $h(2)+f(3)$
(13) $g(f(2))$
(14) $f(h(3))$
(15) $f(h(x))$
(16) $f(h(t))$
(17) $k(5)/g(3)$
(18) $\frac{h(3)}{f(3)}$
(19) $f(2)g(4)$
(20) $h(3)+k(3)$
(21) $h(k(5))$
(22) $f(h(2+t))$

10.2. One-to-One Functions

"until one day... nothing happened"

Gameplan 10.2
(1) *Definition of One-To-one*
(2) *Picture of One-To-one*
(3) *Graph of One-To-one*
(4) *Proof of One-To-one*

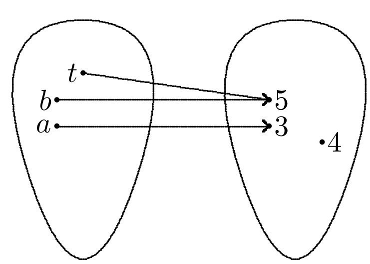
Definitions

(1) A function $f : A \to B$ is called $1-1$ or *one-to-one* iff for every element in $y \in B$ there exists at most one $x \in A$ such that $f(x) = y$. Said another way, every element in the range gets hit *at most once*. Yet another way to put it is that $f(a) = f(b)$ implies $a = b$.
(2) A function $f : A \to B$ is called *onto* iff for every element in $y \in B$ there exists *at least* one $x \in A$ such that $f(x) = y$. Said another way, every element in the range gets hit *at least once*. Yet another way to put it is that $y \in B$ implies there exist $a \in A$ such that $f(a) = y$.
(3) *The Image of* f is by definition the subset of the range that gets hit. In other words,
$$Im(f) = \{y \in B | \text{ there exist } a \in A \text{ with } f(a) = y\}$$
Note that if all elements in the range get hit then the $Im(f) = Range(f)$. Also, note that f is onto iff $Im(f) = Range(f)$.

Examples

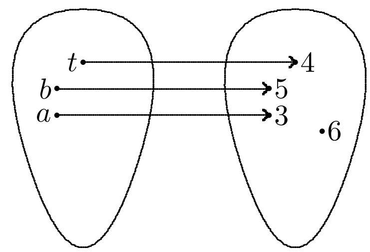
Determine if the function is 1-1, onto, and describe the image of the function.

(1)



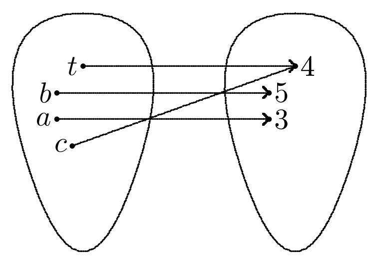
The domain of this function is the set $\{a, b, t\}$. The range is $\{3, 4, 5\}$. The image is $\{3, 5\}$ since these are the only elements that get hit. This function is not 1-1 since 5 in the range gets hit more than one time. This function is not onto since there are some elements in the range that do not get hit.

(2)



The domain of this function is the set $\{a, b, t\}$. The range is $\{3, 4, 5, 6\}$. The image is $\{3, 4, 5\}$ since these are the only elements that get hit. This function is 1-1 since every element in the range gets hit at most one time. This function is not onto since there are some elements in the range that do not get hit.

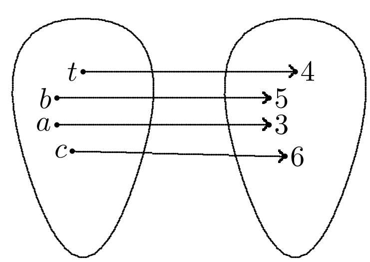
(3)



Solution:

The domain of this function is the set $\{a, b, t, c\}$. The range is $\{3, 4, 5\}$. The image is $\{3, 4, 5\}$ since these are the only elements that get hit. This function is not 1-1 since some element in the range gets hit more than one time. This function is indeed onto since all elements in the range get hit.

(4)



Solution:

The domain of this function is the set $\{a, b, t, c\}$. The range is $\{3, 4, 5, 6\}$. The image is $\{3, 4, 5, 6\}$ since these are the only elements that get hit. This function is 1-1 since every element in the range gets hit at most one time. This function is indeed onto since all elements in the range get hit. When a function is both 1-1 and onto it is called *bijective*.

(5) We now try and algebraic example. Suppose $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$.

Solution:

This function is not 1-1 since some number in the range get hit more more than once for example 4. $f(2) = 4$ and $f(-2) = 4$ thus 4 get hit by -2 and by 2 making the function not 1-1. It is also not onto because the negative numbers in the range will never get hit. This is because the square of a real number will always be non-negative. In fact, the only real numbers that get hit by this function are the non-negative numbers thus the image of this function is $[0, \infty)$. If we let $y = f(x)$ it is sometimes easier to answer all these questions from the graph of the function. To describe the image means to describe which y values get hit. Imagine casting a horizontal shadow of the graph onto the y axis. The shaded y represent the y's that get hit = the image of the function. The question of 1-1 can also be addressed using the graph. To ask how many times elements in the range get hit is the same as asking how many times each y gets hit by an x in the domain. So to check if a function is 1-1, consider a horizontal line through every y value, and check that that y value does not intersect the graph more than once. This insure that no y get's hit by two different x values. This test is commonly known as the *horizontal line test for 1-1.* Of course it presumes you have access to the graph of the function.

(6) Pictures are good for poetry, intuition and feelings, but there is nothing like a good old rigorous, honest, mathematical proof. Prove $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 3x - 1$ is 1-1.

Solution:

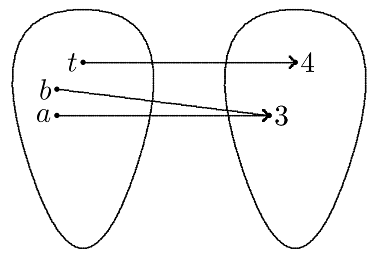
Let $a, b \in \mathbb{R}$ and suppose $f(a) = f(b)$	hypothesis
$3a + 1 = 3b + 1$	definition of f
$3a = 3b$	CLA
$a = b$	CLM

Thus we have proven that if $f(a) = f(b)$ then $a = b$. This is the precise definition of f being 1-1. Take this one to the bank!

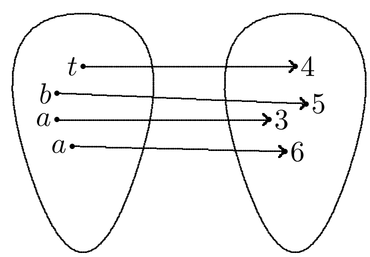
EXCERCISES 10.2

Determine if the following are functions. If so, determine if they are 1-1, onto, what's the image.

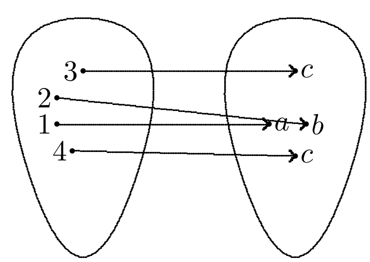
(1)



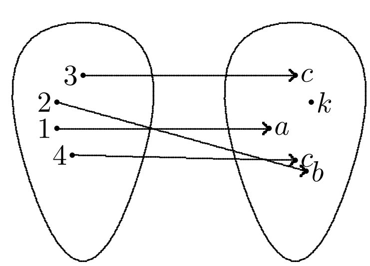
(2)



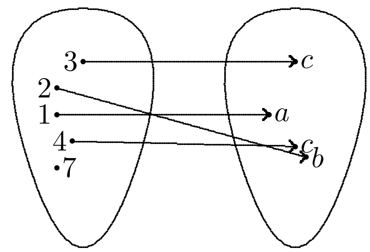
(3)



(4)



(5)



(6) $f : \mathbb{R} \to \mathbb{R}$ with rule $f(x) = \sqrt{x}$
(7) $f : \mathbb{R} \to \mathbb{C}$ with rule $f(x) = \sqrt{x}$
(8) $f : \mathbb{R}^+ \to \mathbb{R}$ with rule $f(x) = \sqrt{x}$
(9) $f : \mathbb{R}^+ \to \mathbb{R}^+$ with rule $f(x) = \sqrt{x}$
(10) $f : \mathbb{Z} \to \mathbb{Z}$ with rule $f(x) = x^2$
(11) $f : \mathbb{Z} \to \mathbb{Z}$ with rule $f(x) =$ The remainder of x when divided by 10
(12) $f : \mathbb{R} \to \mathbb{R}$ with rule $f(x) = 6x + 2$

10.3. Inverse Functions

"until one day... nothing happened"

Gameplan 10.3

(1) *Definition*
(2) *How to Check Inverses*
(3) *Who has Inverse*
(4) *How to find Inverses*

DEFINITION

Recall the ingredients needed to describe inverse pairs. We first need a binary operation in a set. We will operate functions by *composition*. We also need an identity. That is we need a function that does not change any other function when composed by it. This function is $i(x) = x$. At last we are ready to define inverse pairs. If $f : A \to B$ and $g : B \to A$, then f and g are called (composition) inverses if and only if when we composed them together we get the identity.

$$f \circ g(b) = b = i(b) \text{ for all } b \in B$$
$$\text{and}$$
$$g \circ f(a) = a = i(a) \text{ for all } a \in A$$

EXAMPLES

Check to see if the following pairs are inverses.

(1) $f(x) = \frac{1}{2}x$ and $g(x) = 2x$ with both domain/range=$\mathbb{R}$

Solution:

We first check that $f \circ g(x) = x$ for every real number x

$$\begin{aligned} f \circ g(x) &= f(g(x)) && \text{Def of composing} \\ &= f(2x) && \text{def of } g \\ &= \frac{1}{2} \cdot 2x && \text{def of } f \\ &= x && \text{BI} \end{aligned}$$

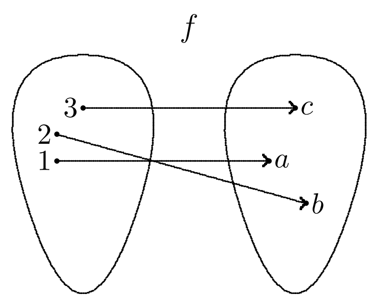
We have check the first part. We now check that $g \circ f(x) = x$ for every real number x

$$
\begin{aligned}
g \circ f(x) &= g(f(x)) && \text{Def of composing} \\
&= g(\frac{1}{2}x) && \text{def of } f \\
&= 2 \cdot \frac{1}{2}x && \text{def of } g \\
&= x && \text{BI}
\end{aligned}
$$

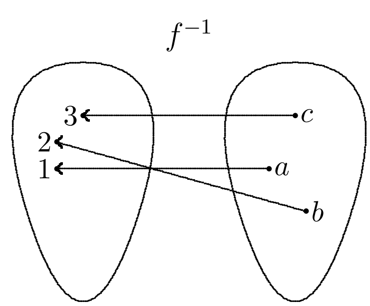
This proves that f and g are in fact inverses of each other.

WHO HAS INVERSE?

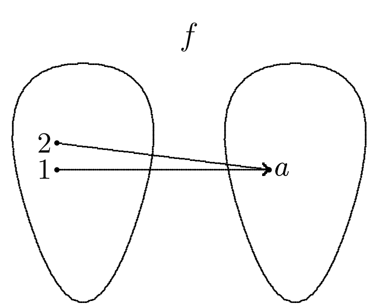
(1)



This function has an inverse. To get the inverse we simply reverse the arrows. The reader should check that composing these functions leaves every element unchanged.

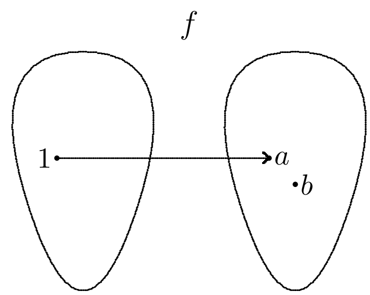


(2)



This function does not have an inverse because if we reverse the arrows a would have two choices to back to 1 or 2. Functions assign a *unique* element to each element in the domain. Thus such a map would not be a function.

(3)



This function does not have an inverse because if we reverse the arrows a would have no choice to back to. Functions assign a unique element to each element in the domain. Thus such a map would not be a function.

The above examples reveal some necessary condition for an inverse to exist, namely that the function be onto and 1-1. Moreover, these are sufficient. We summarize with a theorem:

Theorem

A function has an inverse iff it is 1-1 and onto.

This resolves the question of which functions have inverses. The next logical question is if in fact a function has an inverse, how do we find it. The clever idea is to use the definition of inverses. Observe.

Examples

(1) Find the inverse function for $f(x) = 3x + 4$. Call the inverse function g.

$$\begin{aligned} f(g(x)) &= x && \text{def of inverses} \\ 3g(x) + 4 &= x && \text{def of } f \\ 3g(x) &= x - 4 && \text{CLA} \\ g(x) &= \frac{x-4}{3} && \text{CLM} \end{aligned}$$

(2) Find the inverse function for $f(x) = \frac{1+x}{x}$. Call the inverse function g.

$$
\begin{aligned}
f(g(x)) &= x && \text{def of inverses} \\
\frac{1+g(x)}{g(x)} &= x && \text{def of } f \\
1+g(x) &= xg(x) && \text{CLM} \\
1 &= xg(x)-g(x) && \text{CLA} \\
1 &= g(x)(x-1) && \text{DL} \\
g(x) &= \frac{1}{x-1} && \text{CLM}
\end{aligned}
$$

Excercises 10.3

Check to see if the pairs are inverses of each other. For now, you may assume the corresponding domains and ranges are compatible.

(1) $f(x) = \frac{x-1}{x+1}$ and $g(x) = \frac{x-1}{x+1}$
(2) $f(x) = x^2$ and $g(x) = \sqrt{x}$
(3) $f(x) = x^2 + 1$ and $g(x) = \sqrt{x-1}$
(4) $f(x) = 3x + 5$ and $g(x) = \frac{x-5}{3}$
(5) $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x}$
(6) $f(x) = \frac{1}{x+1}$ and $g(x) = \frac{1}{x-1}$
(7) $f(x) = \frac{x}{x+1}$ and $g(x) = \frac{x-1}{2x}$

Find the inverse if it exist.

(8) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 5x - 3$
(9) $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 3x - 5$
(10) $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(x) = \sqrt{x}$
(11) $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(x) = \sqrt{2x+1}$
(12) $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(x) = \frac{3x-6}{2x+5}$
(13) $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(x) = \frac{4x+2}{x+1}$
(14) $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f(x) = 2x^2 + 4$

10.4. Exponential Functions

"until one day... nothing happened"

Gameplan 10.4
(1) *Definition*
(2) *Examples*
(3) *EFR1-1*
(4) *Solving*

DEFINITION

An exponential function is one where the input variable appears as an exponent. The typical exponential function is:

$$\exp : \mathbb{R} \to \mathbb{R}^{+} \text{ with } \exp(x) = a^x$$

EXAMPLES OF EXPONENTIAL FUNCTIONS

(1) $f(x) = 3^x$
(2) $f(x) = 5^x$
(3) $f(x) = (\frac{2}{-3})^x$
(4) $f(x) = 3^{3x}$
(5) $f(x) = 3^{3x+5}$
(6) $f(x) = 3^{3x^2+7}$
(7) $f(x) = 3^{3x^2+7} + 4x + \pi$
(8) $g(x) = x^x$
(9) $g(x) = x^{x^x}$

PRACTICE EVALUATING

Suppose $f(x) = 2^x$ then:

(1) Evaluate: $f(1)$

$$\begin{aligned} f(1) &= 2^1 && \text{from definition of } f(x) \text{ above} \\ &= 2 && \text{BI} \end{aligned}$$

(2) Evaluate:$f(2)$.

$$\begin{aligned} f(2) &= 2^2 && \text{from definition of } f(x) \text{ above} \\ &= 4 && \text{BI} \end{aligned}$$

(3) Evaluate:$f(t)$.

$$f(t) = 2^t \qquad \text{from definition of } f(x) \text{ above}$$

(4) Evaluate: $f(-3)$.

$$\begin{aligned} f(-3) &= 2^{-3} && \text{from definition of } f(x) \text{ above} \\ &= \frac{1}{2^3} && \text{Def of Negative exponents} \\ &= \frac{1}{8} && \text{BI} \end{aligned}$$

(5) Evaluate: $f(3+2)$.

$$\begin{aligned} f(3+2) &= 2^{3+2} && \text{from definition of } f(x) \text{ above} \\ &= 2^3 \cdot 2^2 && \text{SBE Theorem} \\ &= 8 \cdot 4 && \text{BI} \\ &= 32 && \text{BI} \end{aligned}$$

Graphing Exponential Functions

Note that all the same ideas we learned for other graphs still apply. For example we would still plot points to get acquainted with a graph if it is the very first time we see it. All the shifting reflecting and stretching still works as before (and will always for the next millennium)

Exponential Functions are 1-1 Theorem

If $a \in \mathbb{R}$ then the exponential function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = a^x$ is one to one. (EFR1-1)

Unfortunately, we may not be ready for a proof of this deep theorem. But it may help to convince you that the graphs of these functions satisfy the horizontal line test. If you demand a real proof you have symptoms of a true mathematician and should have yourself examined as soon as possible before it's too late :). Nevertheless, we will reap the fruits of this theorem immediately. Recall that $f(x)$ is 1-1 iff $f(x) = f(y) \implies x = y$. This means if we have a function like $f(x) = 2^x$, it is 1-1. That means if $f(x) = f(y)$ then $x = y$. Said another way (here's the punchline) if $2^x = 2^y$ then $x = y$. This little fact will help us solve many exponential equations. Note it is very important that we consider the domain and range for this function. These exponents must be real to apply the theorem. This theorem is false for complex exponents!!

Solving Some Baby Exponential Equations

(1) Solve $2^x = 8$

Solution:

$$\begin{aligned} 2^x &= 8 && \text{given} \\ 2^x &= 2^3 && \text{def of Expo (right side)} \\ x &= 3 && \text{EFR1-1} \end{aligned}$$

(2) Solve $2^{3x-5} = 8$

Solution:

$$\begin{aligned} 2^{3x-5} &= 8 && \text{given} \\ 2^{3x-5} &= 2^3 && \text{def of Expo (right side)} \\ 3x - 5 &= 3 && \text{EFR1-1} \\ 3x &= 8 && \text{CLA} \\ x &= \frac{8}{3} && \text{CLM} \end{aligned}$$

(3) Solve $3^{2x-3} = 81$

Solution:

$$\begin{aligned} 3^{2x-3} &= 81 && \text{given} \\ 3^{2x-3} &= 3^4 && \text{def of Expo (right side)} \\ 2x - 3 &= 4 && \text{EFR1-1} \\ 2x &= 7 && \text{CLA} \\ x &= \frac{7}{2} && \text{CLM} \end{aligned}$$

Excercises 10.4

(1) Evaluate $f(2)$ in: $f(x) = 3^x$
(2) Evaluate $f(2)$ in: $f(x) = 5^x$
(3) Evaluate $f(2)$ in: $f(x) = (\frac{2}{-3})^x$
(4) Evaluate $f(2)$ in: $f(x) = 3^{3x}$
(5) Evaluate $f(2)$ in: $f(x) = 3^{3x+5}$
(6) Evaluate $f(2)$ in: $f(x) = 3^{3x^2-7}$
(7) Evaluate $f(2)$ in: $f(x) = 3^{3x^2-17} + \pi$
(8) Evaluate $f(2)$ in: $g(x) = x^x$
(9) Evaluate $f(2)$ in: $g(x) = x^{x^x}$
(10) Solve $3^{2x+3} = 81$
(11) Solve $5^{2x+3} = 125$
(12) Solve $2^{2x+3} = \frac{1}{4}$
(13) Solve $3^{2x+3} = \frac{1}{9}$
(14) Solve $10^{2x+3} = 1000000$

(15) Graph $y = 3^x$ (Plot Points!!)

(16) Graph $y = 3^{-x}$ (Use above graph and equation)

10.5. Logarithmic Functions

"until one day... nothing happened"

Gameplan 10.5
(1) *Definition*
(2) *Examples*
(3) *LFR1-1*
(4) *Solving*

The number e

It is one of the most remarkable and underrated number of the universe. It has amazing properties that appear everywhere around us. Functions that describe temperature changes involve e. Formulas that involve money and interest compounding involve e. Population growth models and even the construction of St. Louis Arch involves formulas with e in it. e is a constant roughly equal to 2.7. We can get a more precise estimate by taking a large value for n and plugging it into the formula below. The larger the n the better the estimate. The standard language is that e is defined to be the limit of this sequence as n goes to infinity. We summarize by definition:

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n$$

Definitions of Logs

(1) For $a \in \mathbb{R}^+$ we define $\log_a b$= the power needed to raise b to, to get a. Above a is called the base of the log. An equivalent form of the definition is
$$\log_a b = c \iff a^c = b$$
(2) When the base is omitted, the default value is 10 and it is called a *common log.* That is
$$\log b = \log_{10} b$$
(3) One of the most celebrated bases is e. When the base is e, we use a ln instead of log and we omit the base. In other words, by definition of ln:
$$\ln b = \log_e b$$

As Functions

Assume $a \in \mathbb{R}^+$

The typical Logarithmic function is:

$$\log_a : \mathbb{R}^+ \to \mathbb{R} \text{ with } \log_a(x) = \text{ the power needed on } a \text{ to get } x$$

Practice Computing Logs

(1) Find $\log_3 9$

Solution:

By definition $\log_3 9 =$ the power that 3 needs to get to 9. A moments though reveals that $3^2 = 9$, thus the sought power is 2. Therefore, $\log_3 9 = 2$

(2) Find $\log_3 \frac{1}{3}$

Solution:

By definition $\log_3 9 =$ the power that 3 needs to get to $\frac{1}{3}$. A moments though reveals that $3^{-1} = \frac{1}{3}$, thus the sought power is -1. Therefore, $\log_3 \frac{1}{3} = -1$

(3) So far we only used the first version of the definition of logs. We now use the second version of the definition to find $\log_2 8$. The idea here is that we pretend we don't know what $\log_2 8$ is. What ever it is it's something, so we take the liberty to call it something, we will call it x and assume $x = \log_2 8$. By giving it a name, it will be easier to manipulate it and eventually figure out what $\log_2 8$ is.

Solution:

$$
\begin{aligned}
\log_2 8 &= x && \text{assume (we can name it anything)} \\
2^x &= 8 && \text{def of logs} \\
2^x &= 2^3 && \text{def of Expo} \\
x &= 3 && \text{EFR1-1}
\end{aligned}
$$

So, our unknown quantity $x = \log_2 8 = 3$

(4) Calculate $\log_2 \frac{1}{16}$.

Solution:

$$
\begin{aligned}
\log_2 \frac{1}{16} &= x && \text{assume (we can name it anything)} \\
2^x &= \frac{1}{16} && \text{def of logs} \\
2^x &= 2^{-4} && \text{def of Neg Expo} \\
x &= -4 && \text{EFR1-1}
\end{aligned}
$$

So, our unknown quantity $x = \log_2 \frac{1}{16} = -4$

EXCERCISES 10.5

Simplify.

(1) $\log 100$
(2) $\log 1000$
(3) Estimate $\log 500$
(4) $\log_2 128$
(5) $\log_2 (\frac{1}{128})$
(6) $\log_2 (\frac{1}{8})$
(7) $\log_2 64$
(8) $\log_2 2^5$
(9) $\log_2 2^7$
(10) $\log_2 2^9$
(11) $\log_2 2^{-5}$
(12) $\log_2 2^x$
(13) $\log_3 9 + \log_5 \frac{1}{5}$
(14) $\ln e$
(15) $\ln e^4$
(16) $\ln e^e$
(17) Calculate $\log_3 9$
(18) Calculate $\log_3 10$
(19) Calculate$\log_5 10$
(20) Estimate $\ln 10$
(21) Estimate $\ln 20$
(22) Estimate $\ln 30$
(23) Estimate $\ln \pi$
(24) Estimate $\ln \pi^2$
(25) $\ln e^{blah}$

10.6. Logarithmic Properties

"until one day... nothing happened"

Gameplan 10.6
(1) *Definition*
(2) *Examples*
(3) *Properties*
(4) *Solving*

FAMOUS LOG THEOREMS AND AXIOMS

For all theorems below we assume $a, b, c \in \mathbb{R}^+$. These theorems are also true for other log bases

(1) $\log ab = \log a + \log b$ (Log of Product= Sum of Logs)
(2) for $r \neq 0$, $\log a^r = r \log a$ (nameless theorem)
(3) $\log\left(\frac{a}{b}\right) = \log a - \log b$ (log of Quotient = Difference of Logs)
(4) $\log_a b = \frac{\log_c b}{\log_c a}$ (Change of Base Theorem)
(5) $\log_a(a^x) = x$ (Log and Expo functions are Inverses)
(6) $a^{\log_a x} = x$ (Log and Expo functions are Inverses)
(7) If $A = B$ then $\log A = \log B$ (Cancellation Law of Logs (axiom))
(8) If $C = B$ then $a^C = a^B$ (Cancellation Law of Exponents (axiom))

Proofs:

The proof of these theorems are well within our current technology, and left as an important exercise to the student. For now we turn our attention to the beautiful art of solving exponential and logarithmic equations

EXAMPLES

(1) Solve $2^x = 10$

Solution:

$$
\begin{aligned}
2^x &= 10 && \text{given}\\
\log 2^x &= \log 10 && \text{CLL}\\
x\log 2 &= \log 10 && \text{Nameless Thm}\\
x(\log 2)\cdot\frac{1}{(\log 2)} &= (\log 10)\frac{1}{(\log 2)} && \text{CLM}\\
x &= \frac{(\log 10)}{(\log 2)} && \text{B.I.} \qquad (5)
\end{aligned}
$$

note: by step 5 we have indeed solved for x since x is isolated on one side.

(2) Solve $2^{(3x-4)} = 10\cdot 5^x$

Solution:

$$
\begin{aligned}
2^{(3x-4)} &= 10\cdot 5^x && \text{given}\\
\log 2^{(3x-4)} &= \log(10\cdot 5^x) && \text{CLL}\\
\log 2^{(3x-4)} &= \log 10 + \log 5^x && \text{LP=SL}\\
(3x-4)\log 2 &= \log 10 + x\log 5 && \text{Nameless Thm}\\
3x\log 2 - 4\log 2 &= \log 10 + x\log 5 && \text{DL (note now it's just linear eq!)}\\
3x\log 2 - x\log 5 &= \log 10 + 4\log 2 && \text{CLA}\\
x(3\log 2 - \log 5) &= \log 10 + 4\log 2 && \text{DL}\\
x &= \frac{\log 10 + 4\log 2}{3\log 2 - \log 5} && \text{CLM}
\end{aligned}
$$

note: You can now take this one to the bank!!

(3) $\log_3(x-1) = 2 + \log_3(x+2)$

Solution:

$$\log_3(x-1) = 2 + \log_3(x+2) \qquad \text{given}$$

(2) $$\log_3(x-1) - \log_3(x+2) = 2 \qquad \text{CLA}$$

(3) $$\log_3(\frac{x-1}{x+2}) = 2 \qquad \text{LQ=DL}$$

$$3^2 = \frac{x-1}{x+2} \qquad \text{Def of Logs}$$

$$9 = \frac{x-1}{x+2} \qquad \text{BI}$$

(6) $$9(x+2) = x-1 \qquad \text{CLM}$$

$$9x + 18 = x - 1 \qquad \text{DL}$$

$$8x = -19 \qquad \text{CLA}$$

$$x = \frac{-19}{8} \qquad \text{CLM}$$

Note that step (2), CLA hold when when we add a real number to both sides, here we are adding an unknown quantity $-\log_3(x+2)$ which may or may not be real. Step (3) also depends on the condition needed for LQ=DL, that is that the stuff inside the log be positive and real. There is a similar assumption for step (6). The moral of the story is that we must check our answer to see if in fact we've solved the equation. Which we haven't in this case because as we plug in $x = \frac{-19}{8}$ into the first part of the equation $\log_3 -19/8 - 1$ get an undefined quantity since the log function is defined only for $x \in \mathbb{R}^+$

(4) Solve $\ln x + \ln(x-4) = \ln(2x-3)$

Solution:

$$\ln x + \ln(x-4) = \ln(2x-3) \qquad \text{given}$$

$$\ln(x)(x-4) = \ln(2x-3) \qquad \text{LP=SL}$$

$$e^{\ln(x)(x-4)} = e^{\ln(2x-3)} \qquad \text{CL Expo}$$

$$(x)(x-4) = (2x-3) \qquad \text{Same Base Inverse Functions}$$

$$x^2 - 4x = 2x - 3 \qquad \text{DL}$$

$$x^2 - 4x + 2x + 3 = 0 \qquad \text{CLA}$$

$$x^2 - 2x - 3 = 0 \qquad \text{BI}$$

$$(x-3)(x+1) = 0 \qquad \text{BI}$$

$$x = 3 \text{ or } x = -1 \qquad \text{ZFT, BI}$$

We check each of these solution and determine that $x = 3$ is the only one that works!

Excercises 10.6

(1) $3^x = 9$
(2) $3^{2x} = 9$
(3) $3^{2x+6} = 9$
(4) $3^{2x-5} = 81$
(5) $3^{2x-5} = 81 \cdot 3^x$
(6) $(\frac{1}{9})^{2x-5} = 81$
(7) $3^{2x} - 3^x = -2$
(8) $9^x - 5(3^x) = -6$
(9) $3^{2x+6} = 2 \cdot 3^{x-3}$
(10) $3^{2x+6} = 5^{x-3}$
(11) $3^{2x+6} = 5^{x-3}$
(12) $e^{2x} - e^x - 6 = 0$
(13) $14e^{3x+2} = 560$
(14) $\log_3(3x+4) = 2$
(15) $\log_3(3x+4) = -3$
(16) $\log_3(3x+4) = 0$
(17) $\log_3(3x+4) = \log_3(2x-3)$
(18) $\log_3(3x+4) = \log_3(2x-5)$
(19) $2\log_3(3x+4) = \log_3(52x-3)$
(20) $\log_4(3x-5) = 3$
(21) $\log_2(x+3) + \log_2(x-3) = 4$
(22) $\log_3(x-1) = 2 + \log_3(x+2)$
(23) $\ln x + \ln(x-4) = \ln(2x-5)$
(24) $\log(x+7) - \log(x+2) = \log(x+1)$
(25) $\log_3 x + \log(x-8) = 2$
(26) $\log_4 x + \log_4(x+7) = 1$
(27) $\ln\sqrt{x+1} = 2$
(28) Simplify $\ln(\log_5(3^{\log_3 5^{e^3}}))$
(29) $81^{x-1} = 27^{2x}$
(30) $3^{3x-2} = 24$
(31) $3\log_2 x = -\log_2 27$
(32) $e^{x+3} = \pi^x$
(33) $\log_{16} x + \log_4 4 + \log_2 x = 7$
(34) $3x - \pi = 7x + e$
(35) $x^2 - (\log 5)x = \pi^3$
(36) $81^{x-1} = 27^{2x}$
(37) $3^{3x-2} = 24$
(38) $\log_9(4x) - \log_9(x-3) + \log_9(x^3-27) = 2\log_9(\sqrt{x})$
(39) $\log_3 x + \log_3(x-8) = 2$
(40) $\log_4 x + \log_4(x+7) = 1$
(41) $\log 5x^2 - \log 3x + 4 = 3$
(42) $\log(5x)^2 - \log 3x + 4 = 3$
(43) $(\log_3 x)^2 + 5\log_3 x = 14$

10.7. Applications

"until one day... nothing happened"

Gameplan 10.7
(1) *Idea*
(2) *Examples*
(3) *Properties*
(4) *Solving*

The Idea

Here we get a chance to apply some of our equation-solving skill to solve some real life problems. The idea is to read the problem and try to find hints that turn into equation. If all goes well, these equations will turn out to be exponential and/or logarithmic equations that we can solve. If there is more than one unknown you may need more that one equation. Generally, if you have two unknowns you may hope to find a solution by looking for two equations that will make up a system of two equations and two unknowns. Then you can use all the methods for solving non-linear systems that we used in the previous chapter.

Some Function Models from Real Life

(1) *Money*
  (a) Given principal (initial amount) of P dollars placed in bank earning an annual interest rate r compounded n times a year, then the amount of money in this account after t years is given by
$$A(t) = P(1 + \frac{r}{n})^{nt}$$
  (b) If the interest rate is compounded continuously then the Amount after t years is given by (e appears everywhere)
$$A(t) = Pe^{rt}$$
(2) *Population*
  (a) Given an initial population of P_0 with reproduction rate k (k depends on the species) the Population after t years is given by (e appears everywhere)
$$P(t) = P_0 e^{kt}$$
(3) *Radio Active Decay*
  (a) Given an initial Amount of radioactive substance of A_0 with decay rate k (k depends on the substance) the Amount Present after t years is given by (e

appears everywhere)

$$A(t) = A_0 e^{rt}$$

(4) *Newton's Cooling Law*

(a) Given an initial Temperature of T_0 with cooling rate k (k depends on the the body) and Ambient temperature A, the Temperature after t hours is given by (e appears everywhere)

$$T(t) = A + (T_0 - A)e^{kt}$$

EXAMPLES

(1) Compounded 4 times a year at 12% annual rate, how much money would there be in an account after 5 years, if we open the account with \$100?

Solution:

Note: $P = \$100$, $n = 4$, $r = .12$, and $t = 5$ thus..

$A(t) = P(1 + \frac{r}{n})^{nt}$	given money model
$A(5) = \$100(1 + \frac{.12}{4})^{4 \cdot 5}$	given
$A(5) = \$100(1.03)^{20}$	BI
$A(5) = \$100(1.80)$	BI
$A(5) = \$180$	BI

(2) same as above but this time compound the interest continuously.

Solution:

we still have: $P = \$100$, $r = .12$, and $t = 5$ thus..

$A(t) = Pe^{.12r}$	given money model
$A(5) = \$100e^{.12 \cdot 5}$	given
$A(5) = \$100e^{.6}$	BI
$A(5) = \$100(1.82)$	BI
$A(5) = \$182$	BI

(3) Suppose I have 5 rabbits and suppose they double every month. Approximately, how many months until I have 10000 rabbits?

Solution:

note: we want to solve for t such that $P(t) = 10000$.That is we want to find t such $10000 = P_0 e^{kt}$. We could solve for t here but we have several other variables. We would like to find a numerical solution to t that tells us how many months. So we start to see which other variables we know. First, we also know $P_0 = 5$. That reduces our equation to finding t such that $10000 = 5e^{kt}$ which can further be reduced to $2000 = e^{kt}$ (CLM) but that is still one variable too many. We don't know what the growth rate k is. However, we do know the amount of rabbits double every 2 months so, we know $P(2) = 2P_0$. Which becomes the equation $P_0 e^{2k} = 2P_0$. This can then be reduced to $e^{2k} = 2$ by (CLM). We can not think of this as a system of two equations two unknowns, and we solve it using the methods we learned before. In this case, we will first solve for k on the second equation and then plug back the value of k into the first equation.

(1)	$2000 = e^{kt}$	see above
(2)	$e^{2k} = 2$	see above
(3)	$2k = \ln 2$	Def of Logs
(4)	$k = \frac{\ln 2}{2}$	CLM
(5)	$\ln 2000 = kt$	Def of Log on (1)
(6)	$t = \frac{\ln 2000}{k}$	CLM
(7)	$t = \frac{\ln 2000}{\frac{\ln 2}{2}}$	Sub (4) into (6)
(8)	$t = \frac{2 \ln 2000}{\ln 2}$	BI
(9)	$t \approx 21.9$months	Calculator

EXCERCISES 10.7

(1) *Money* Suppose you have 2000 in the stock market. How long will it take to double your money assuming a 10% annual return compounded continuously.
  (a) how much will you have after 5 years?
  (b) how much will you have after 5 years if the interest rate is doubled.
  (c) how much will you have after 30 years if the interest rate is 15%.
  (d) At 7% how long does it take to double your money?
  (e) At 14% how long does it take to double your money?

(2) *Radio Active Decay* Suppose a local nuclear plant has an accident close to your house and there is 2000 grams of radioactive isotopes released into the environment. Suppose the isotopes have a half life of 5 years.

(a) Use the fact about the half life to find the decay rate constant.
(b) How much radio active substance will be left after 23 years?
(c) How long until there is only .7 grams left?

(3) *Radio Active Decay* Suppose the we know that every kilogram of live dinosaur has 3 grams of a particular type of carbon molecule that starts decaying once the dinosaur dies with a half life of 1000 years.
(a) Use the fact about the half life to find the decay rate constant.
(b) Suppose a kilogram of dinosaur is found in Balboa Park. After examining it, it is determined that there is .0035 grams of the carbon substance left. How long ago did it die?

(4) *Temperature- Newton's Cooling Law* Suppose, I put 70° wine in my 40° refrigerator. Suppose, after 1 hour I check it and it is 65°.
(a) Use the time after 1 hours to find the cooling constant
(b) What will the temperature be after 2 hours, 3 hours?
(c) How long until the wine is 45°

CHAPTER 11

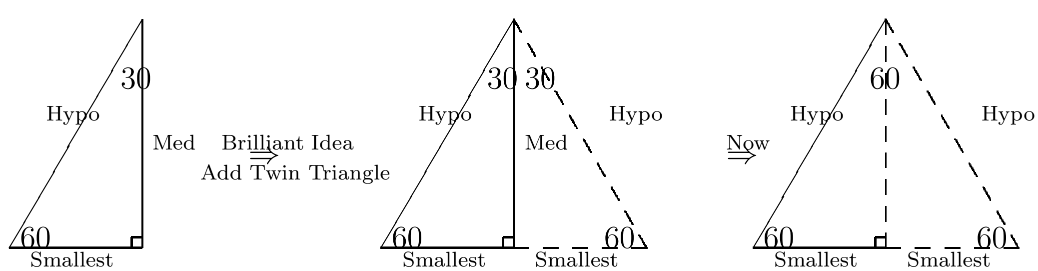
Introduction to Trigonometry

11.1. Famous Triangles

30-60 Triangles

30-60 triangles also enjoy a very place in every scientist's heart. First, they are right triangles, since one angle is 30, another 60, the last angle must e 90 degrees. Thus, they enjoy all the benefits of a right triangle, i.e. Pythagoras Theorem. In addition, their sides have another very special relationship. We will explore this relationship by adding a twin triangle next to it. Observe:

Typical 30-60 Triangle:



And now.... we let the figuring begin!!!! How does the small side compare to the hypotenuse? The last triangle is an Equilateral triangle since all its angles are equal. Therefore, all sides are equal. Therefore, we get the very special relationship for 30-60 triangles:

The small side is half the hypotenuse

But we are not done. For ease of writing we will call the sides h for the hypotheses, m for the medium side *which is always the one across from the* $60°$, and we will use s for the smallest side which is always the one across the smallest angle, $30°$. Now, we resume to the figuring. Concentrating on the original triangle we use our new-found knowledge that $h = 2s$ (hypotenuse is twice the small side). What about the medium side? We apply the powerful Pythagoras Theorem to conclude...

$h^2 = s^2 + m^2$	P.T.
$(2s)^2 = s^2 + m^2$	Substitute
$(2s)(2s) = s^2 + m^2$	Def of Expo
$4s^2 = s^2 + m^2$	Def of Expo
$3s^2 = m^2$	C.L.A
$m = \pm\sqrt{3s^2}$	SRP
$m = \sqrt{3s^2}$	m is a Positive length
$m = \sqrt{s^2}\sqrt{3}$	Sqrt of Prod = Prod of Sqrt*
$m = s\sqrt{3}$	Def of Sqrt

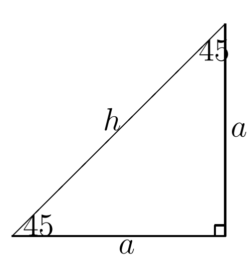
Thus we get another very important relationship. Namely,

The medium side is $\sqrt{3}$ times the smallest side

These relationships amongst the sides of a right triangle are very useful in that once we know *any* one of the sides, we can easily figure out the rest of the sides. That is it! all it takes is one side, and the rest is Duck Soup!!!

45-45 Triangles

Ultimately, we get the same conclusion for the 45-45 triangles. That is, given any one of the sides we can immediately determine the other two sides. However, the actual figuring out of the sides will be nowhere near as exiting as the case was for 30-60 triangles. For starters, the 45-45 triangles are isoseles. Two of the angles are equal thus the two corresponding sides are equal. So we have:
Typical 45-45 Triangle:



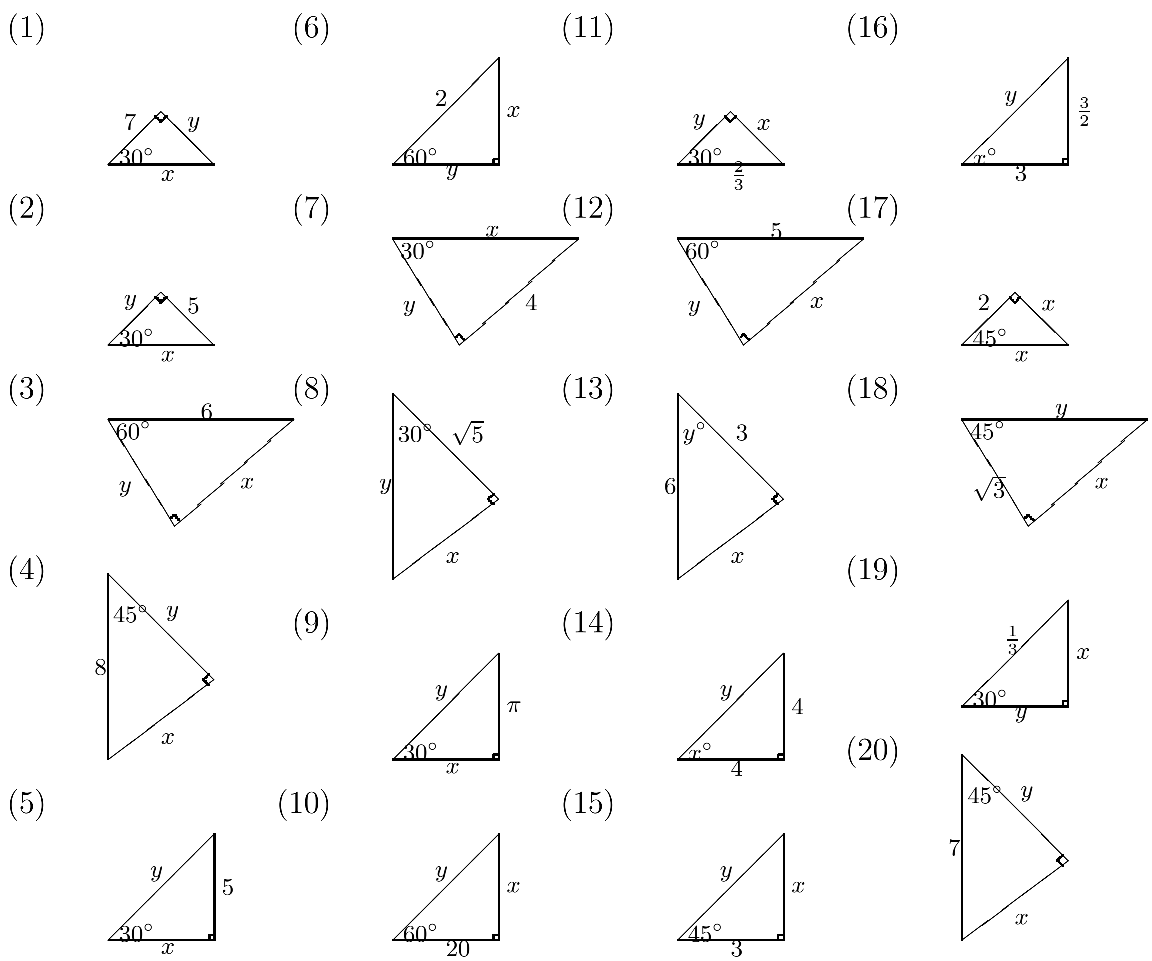
We invoke Pythagoras....

$$
\begin{aligned}
h^2 &= a^2 + a^2 && \text{P.T.} \\
h^2 &= 2a^2 && \text{B.I.} \\
h &= \pm\sqrt{2a^2} && \text{SRP} \\
h &= \sqrt{2a^2} && h \text{ is positive} \\
h &= a\sqrt{2} && \text{B.I.}
\end{aligned}
$$

That it! This tells us all we need to know. The two side are equal and to get the hypotenuse we just multiply by $\sqrt{2}$. Again, with any one of the sides we can determine the rest of the sides.

Excercises 11.1

Figure out all sides of the triangles.



11.2. Trigonometry Functions

Define The Trigonometry Functions

(1) Domain for sin and cos = the set of all possible angle measurement. (for other trig functions the domain has to be restricted)
(2) Range = the real numbers in interval $[-1, 1]$
(3) The unambiguous rule is as follows:
   (a) Given angle θ, draw the angle in standard position on the xy-plane using a segment of any finite length.
   (b) Make a triangle by dropping a perpendicular from the endpoint of the line segment *to the x-axis**.
   (c) Determine all the lengths of the sides of the triangle, including appropriate signs.
   (d) From the origin, determine which side is the adjacent, opposite and the hypotenuse.
   (e) Then the famous trig functions are defined as follows:

$$\cos\theta = \frac{adj}{hyp} \qquad \sin\theta = \frac{opp}{hyp} \qquad \tan\theta = \frac{opp}{adj}$$

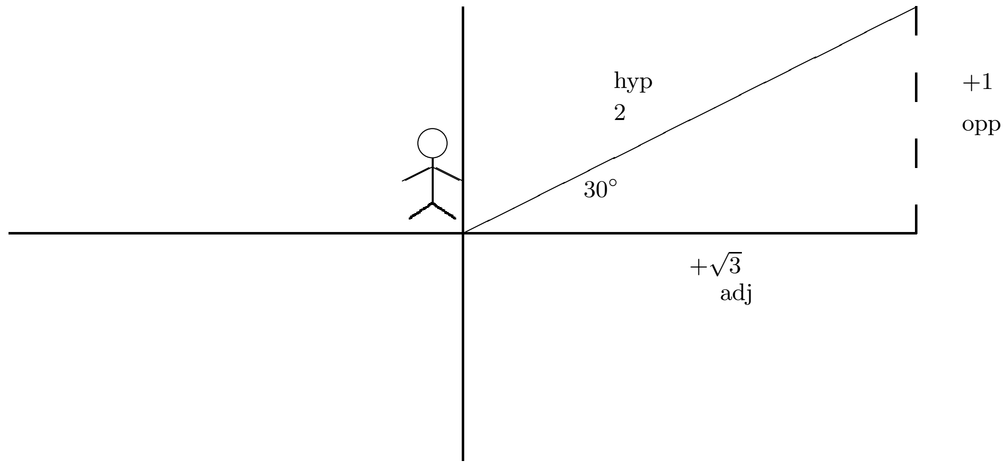
$$\sec\theta = \frac{hyp}{adj} \qquad \csc\theta = \frac{hyp}{opp} \qquad \cot\theta = \frac{adj}{opp}$$

Examples, Evaluating Trig Functions

(1) find $\sin 30°$

Solution:

We first draw a segment from the origin $30°$ counterclockwise from the positive $x - axis$. We will choose a segment of length 2. Then we drop a perpendicular to the x axis, and label all sides including positive or negative signs. The picture should look like:



Once we have our picture the rest is easy. $\sin 30° = \frac{opp}{hyp} = \frac{1}{2}$ In fact we can figure out all trig functions at 30°.

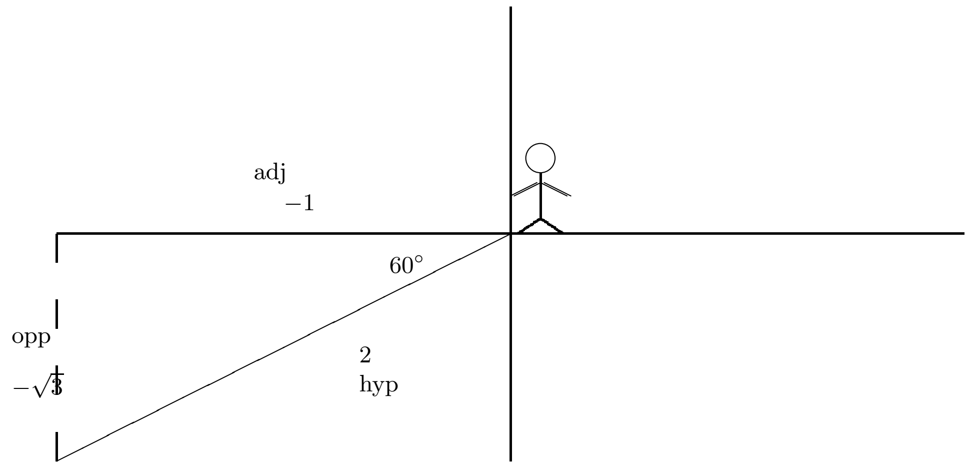
$$\cos 30° = \frac{\sqrt{3}}{2} \qquad \sin 30° = \frac{1}{2} \qquad \tan 30° = \frac{1}{\sqrt{3}}$$

$$\sec 30° = \frac{2}{\sqrt{3}} \qquad \csc 30° = \frac{2}{1} \qquad \cot \theta = \frac{\sqrt{3}}{1}$$

(2) Without calculators. Evaluate all trig functions at 240°

Solution:

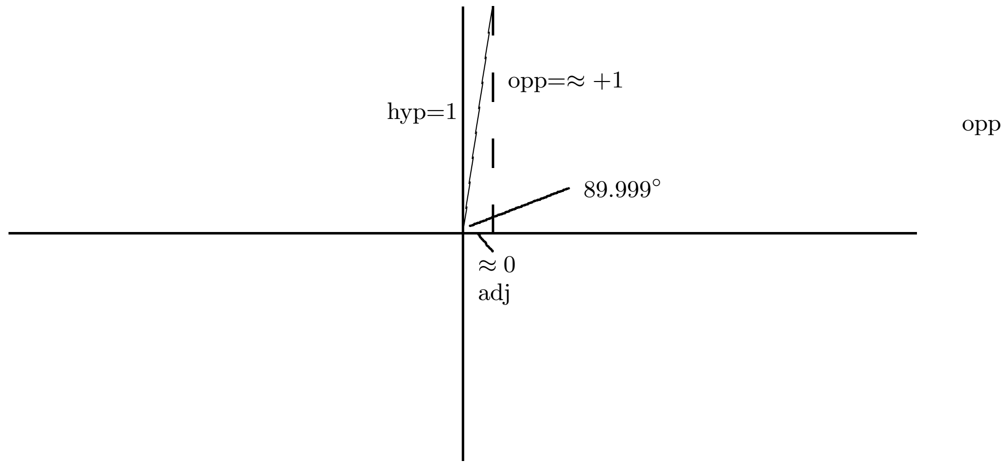
We first make our drawing, again we choose a segment of length 2, label all sided including signs. Again we go from the positive x axis counterclockwise for 240°. That is 60° past 180°. Therefore the reference angles is in the 4th quadrant with 60°. The picture should look like this. And the rest of the problem is easy.



$$\cos 240^\circ = \frac{-1}{2} \qquad \sin 240^\circ = \frac{-\sqrt{3}}{2} \qquad \tan 240^\circ = \frac{-\sqrt{3}}{-1}$$

$$\sec 240^\circ = \frac{2}{-1} \qquad \csc 240^\circ = \frac{2}{-\sqrt{3}} \qquad \cot 240^\circ = \frac{-1}{-\sqrt{3}}$$

(3) Our last example will reveal what the asterisk on the definition of the function is all about. The problem is that sometime you may not be able to *drop a perpendicular to the x-axis to make a triangle.* This happens if the angle ends up on one of the axis. 90° is a perfect example. You draw your segment it ends up right on the positive $y-$axis then we can't make a triangle. Unless, we accept flat triangles. That is triangles where one of the sides has length 0. Indeed, we will resolve this issue by allowing ourselves the liberty of calling these figures triangles. Or if this is too radical of an idea we can use a triangle that is incredibly close to 90° say 89.99999° and carry on as usual. Here the picture for 90°. This time we will use a segment of length 1.



now we're in business...

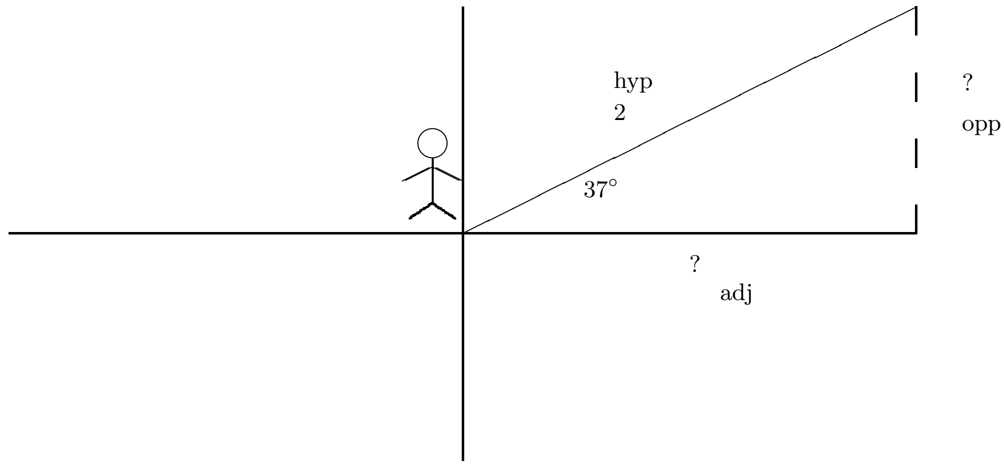
$$\cos 90^\circ = \frac{0}{1} = 0 \qquad \sin 90^\circ = \frac{1}{1} = 1 \qquad \tan 90^\circ = \frac{1}{0} = \text{ undefined}$$

$$\sec 90^\circ = \frac{1}{0} = \text{ undefined} \qquad \csc 90^\circ = \frac{1}{1} = 1 \qquad \cot 90 = \frac{0}{1} = 0$$

(4) find $\sin 37^\circ$

Solution:

we proceed to draw the the segment, this time of length 2 and label all sides... we get..



the problem here is that we don't know how to figure out the rest of the sides, and it is here that we really begin to appreciate famous triangles 30-60, and 45-45, because for such triangles one side was enough, with one side we could figure all other sides. No such luck here, and that is why 37-53 triangles are next to dirt as far as beauty and fame goes. Nevertheless, all hope is not lost. We resort to our calculators for a bit of grunt work. Make sure your calculator is set to degrees and punch in $\sin 37° = .602$ (you may have to type 3 7 sin= depending on your calculator). Incidentally, in the old days people would carry huge textbooks full of these tables to look up these values. Other carried slide-rulers that also have good approximation of these values.

EXCERCISES 11.2

Use calculators ONLY for non-famous terminal angles.

(1) $\sin 30°$
(2) $\cos -30°$
(3) $\tan 300°$
(4) $\sin -60°$
(5) $\cos 180°$
(6) $\sin 180°$
(7) $\cot 180°$
(8) $\sin 225°$
(9) $\cos -120°$
(10) $\tan -300°$
(11) $\sin -45°$
(12) $\sin 315°$
(13) $\sin 675°$
(14) $\cos -330°$
(15) $\sec 315°$
(16) $\csc 675°$
(17) $\cot -330°$
(18) $\sin \pi$
(19) $\cos \frac{2\pi}{3}$
(20) $\tan \frac{4\pi}{7}$
(21) $\sin \frac{4\pi}{7}$
(22) $\cos \frac{-5\pi}{7}$
(23) $\sin \frac{\pi}{2}$
(24) $\cos \frac{3\pi}{2}$
(25) $\tan -\pi$
(26) $\cos \frac{17\pi}{3}$
(27) $\tan \frac{4\pi}{6}$
(28) $\sin \frac{4\pi}{4}$
(29) $\cos \frac{-5\pi}{6}$
(30) *$\cos 1°$
(31) *$\cos 1$
(32) $\sin 31°$
(33) $\cos -33°$
(34) $\tan 235°$
(35) $\sin 15°$
(36) $\cos 20°$
(37) If $\sin\theta = \frac{3}{5}$ find $\cos\theta$
(38) If $\sin\theta = \frac{3}{5}$ find $\cot\theta$

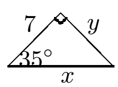
11.3. Non-Famous Triangles

Trig to Solve Non-Famous Triangles

To solve a triangles means to figure out the measurements of all of its 3 sides and 3 angles. We've conquered famous triangles. We now conquer non-famous triangle using our trig function and calculators. We will approximate to 2 decimal values.

Examples

(1)



Solution:

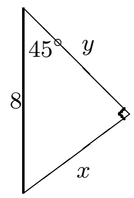
standing at the known angle the tan function says $\tan 35° = \frac{opp}{adj} = \frac{y}{7}$ so we get...

$$
\begin{aligned}
\tan 35° &= \frac{y}{7} && \text{def of tan}\\
7\tan 35° &= y && \text{CLM}\\
y &= 7\tan 35° && \text{Reflexive}\\
y &= 7(.70) && \text{Calc}\\
y &= 4.9 && \text{BI}\\
y^2 + 7^2 &= x^2 && \text{Pyth}\\
(4.9)^2 + 7^2 &= x^2 && \text{Sub}\\
73.1 &= x^2 && \text{BI}\\
\pm\sqrt{73.1} &= x && \text{SRP}\\
8.54 &= x && \text{positive length side}
\end{aligned}
$$

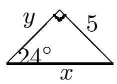
thus we have solve for both $x = 8.54$ and $y = 4.9$

Excercises 11.3

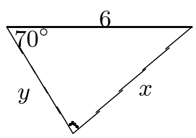
(1)



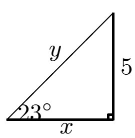
(2)



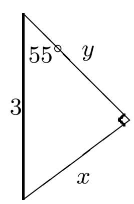
(3)



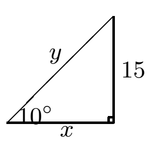
(4)



(5)



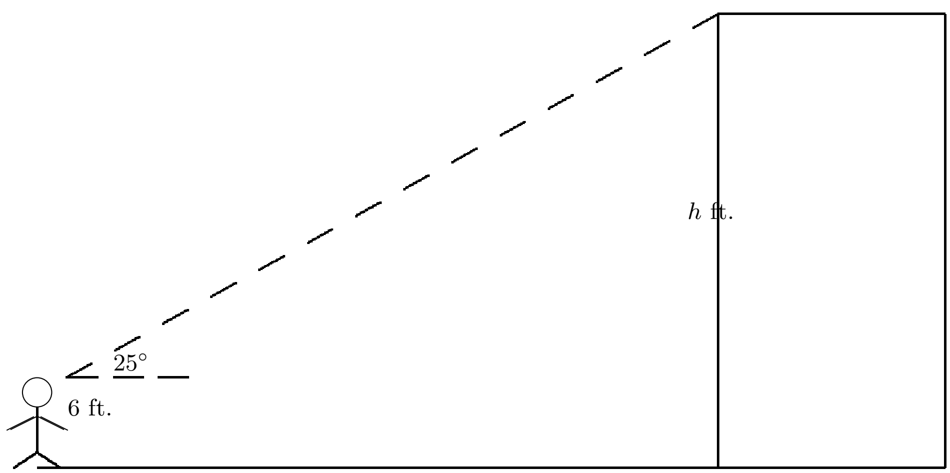
(6)



11.4. Non-Famous Triangles

EXAMPLES

(1) *Measuring the building with a protractor.*

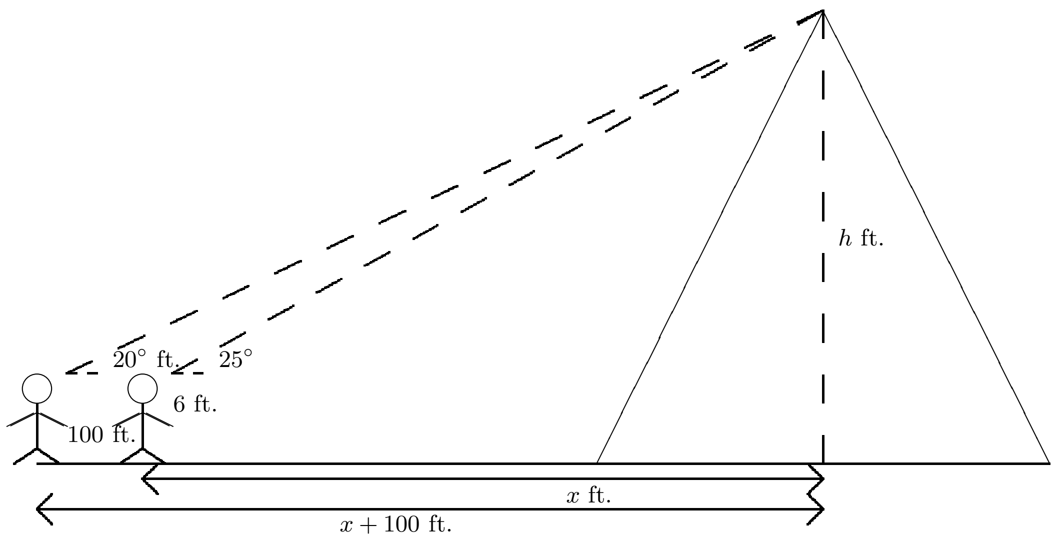


Solution:

$$\begin{aligned} \tan 25^\circ &= \frac{h-6}{50} && \text{def of tan} \\ 50\tan 25^\circ &= h-6 && \text{CLM} \\ 50\tan 25^\circ + 6 &= h && \text{CLA} \\ 50(.47)+6 &= h && \text{Calc} \\ 29.6 &= h && \text{BI} \end{aligned}$$

You can take this one to the bank!!!

(2) This is one of my favorite problems. Say you are walking around the pyramids of Egypt and you want to figure out how tall they are. You glance up at the top and read off the angle, take a few steps (10 ft.) and measure the angle again and wham! it's all over just from these two reading you should be able to determine the height. keep in mind there was no need to even come near the pyramids. Just noodle power.



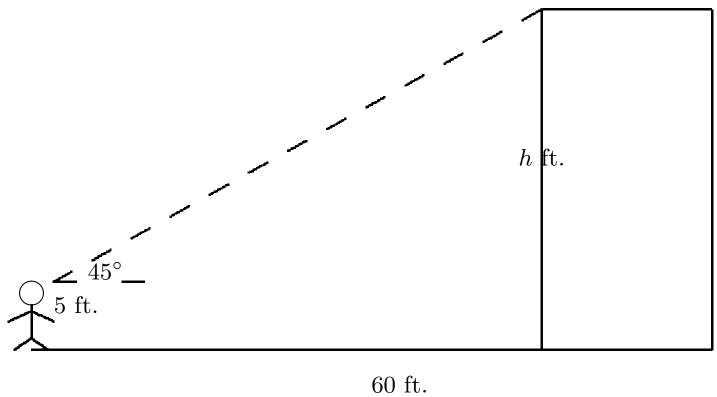
Solution:

(1)	$\tan 25^\circ = h - 6x$	def of tan
(2)	$\tan 20^\circ = h - 6x + 100$	def of tan
(3)	$.46 = \frac{h-6}{x}$	Calc
(4)	$.36 = \frac{h-6}{x+100}$	Calc (note now eq (3) and (4) are just a 2×2 system)
(5)	$.46x = h - 6$	CLM on(3)
(6)	$x = \frac{h-6}{.46}$	CLM
(7)	$x = \frac{h-6}{.36} - 100$	CLM & CLA on (4)
(8)	$\frac{h-6}{.36} - 100 = \frac{h-6}{.46}$	transitivity eq (6) and (7) note we killed x :)
(9)	$\frac{h}{.36} - \frac{6}{.36} - 100 = \frac{h}{.46} - \frac{6}{.46}$	ATT
(10)	$\frac{h}{.36} - \frac{h}{.46} = \frac{6}{.36} - \frac{6}{.46} + 100$	CLA
(11)	$(.60)h = 103.623$	Calc, DL
(12)	$h = 166.623 ft.$	Calc, DL

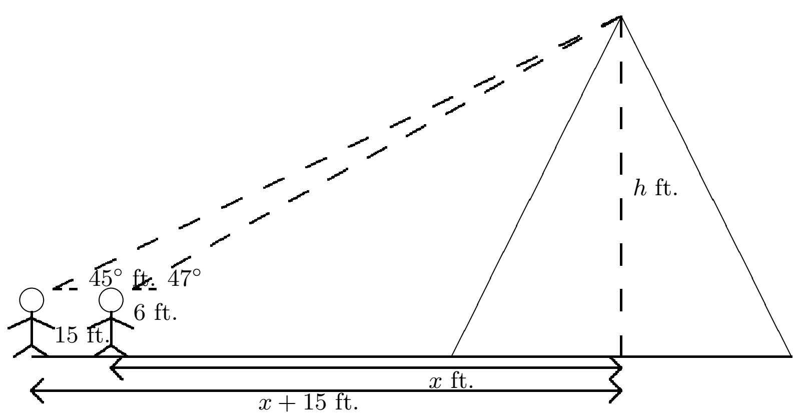
Excellent!!

Excercises 11.4

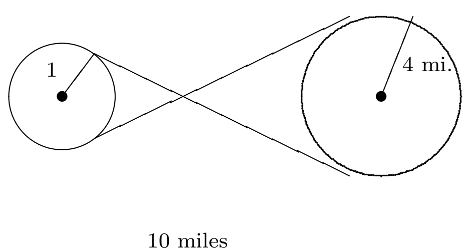
(1) *Measuring the building with a protractor.*



(2) Find the height h



(3) how long is this belt?



CHAPTER 12

Sequences and Series

12.1. Sequence

What is..

A sequences is a bunch of ordered numbers. These numbers are often called terms. The first of these terms is often called a_1 while the second term is called a_2. In general, the nth term is represented by a_n.

Ways of describing sequences

(1) By listing a few terms, until the pattern is obvious...
   (a) $1, 4, 9, 16, 25, 36, \ldots$ this is the sequence of squares...
   (b) $0, 3, 6, 9, \ldots$ this is the sequence of multiples of 3...
   (c) $0, 3, 8, 24, 35, \ldots$ this is the sequence of squares minus one...
   (d) $1, 2, 4, 8, 16, 32, 64, \ldots$ this is the sequence of powers of 2.
   (e) $1, 1, 2, 3, 5, 8, 12, \ldots$ this a famous sequence (Fibonacci), each term is derived by adding the previous two terms.
(2) By Giving a recipe for the nth term, a_n (in terms of n and constants). This is usually called the closed form of the sequence. It is often the most desirable becuase it makes it easy to compute any term of the sequence. Using this method, the above sequences would be described as follows.
   (a) $a_n = n^2$
   (b) $a_n = 3n - 3$
   (c) $a_n = n^2 - 1$
   (d) $a_n = 2^{n-1}$
   (e) The Fibonacci sequence requires a little more thinking to get the formula for the nth term.
(3) By Describing how to go from one term to the very next one. When sequences are described this way they are called *recursive sequences*, or it's said that the sequence is defined *recursively*. Again, we take the same sequences as above...
   (a) $a_n = a_{n-1} + 2n + 1$ with $a_1 = 1$
   (b) $a_n = a_{n-1} + 3$ with $a_1 = 1$
   (c) $a_n = a_{n-1} + 2n - 1$ with $a_1 = 1$
   (d) $a_n = 2a_{n-1}$ with $a_1 = 1$
   (e) $a_n = a_{n-1} + a_{n-2}$ with $a_1 = a_2 = 1$

Arithmetic Sequences

(1) *Arithmetic Sequences* are sequences where all consecutive terms have a common difference.
(2) The recursive form of every arithmetic sequence looks like:

$$a_n = a_{n-1} + d$$

where d is the common difference.
(3) The closed form of every arithmetic sequence looks like

$$a_n = dn + stuff$$

where the stuff may have to be adjusted by some constant to make sure the sequence starts on the first term.
(4) (examples) Write $3, 8, 13, 18, \ldots$ in recursive form.

Solution:

we immediately recognize this is an arithmetic sequence because all consecutive terms differ by 5. So the recursive form would be in words, "start with 3 and to go to the next term take the previous one and add 5". Algebraically, $a_1 = 3$ and $a_n = a_{n-1} + 5$
(5) (examples) Write $3, 8, 13, 18, \ldots$ in closed form.

Solution:

again, we know this is an arithmetic sequence increasing by 5's so the closed form looks like $a_n = 5n + stuff$. Moreover, we want to start with the first term 3. So, $a_1 = 3 = 5 \cdot 1 + stuff$. This implies that the *stuff* must be -2. We conclude $a_n = 5n - 2$ and verify this gives the right terms.

Geometric Sequences

(1) Geometric sequences are sequences where all consecutive terms have a common quotient.
(2) The recursive form of a geometric sequence is to always take the previous term and multiply by the common ration r. Thus the closed form of a geometric sequence will always be $a_n = ra_{n-1}$
(3) The closed form will always be $a_n = (stuff)r^n$ where the *stuff* is a needed factor to get the sequence started on the right term.
(4) (example) Write $5, 10, 20, 40, 80, 160, \ldots$ in recursive form.

Solution:

first we note this is indeed a geometric sequence with a common ratio of 2. To get the next term we always multiply by 2. Thus the recursive form of the sequence is $a_1 = 5$ and $a_n = 2a_{n-1}$

(5) (example) Write $5, 10, 20, 40, 80, 160, \ldots$ in closed form.

Solution:

Since we know this is a geometric sequence with ratio 2, the closed form looks like $a_n = (stuff)2^n$. The first term has to be 5 so $a_1 = 5 = (stuff)2^1$ then it has to be that $stuff = \frac{5}{2}$ and our closed form of the sequence is $a_n = (\frac{5}{2})2^n$.

EXCERCISES 12.1

Rewrite the following sequences in closed form, in recursive form, and by listing the first 5 terms. Of course one of the forms is already given, find the other two.

(1) $1, 6, 11, 21, 26, \ldots$
(2) $-5, -8, -11, -14, \ldots$
(3) $-2, 4, -8, 16, -32$
(4) $5, 5, 5, 5, 5, \ldots$
(5) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$
(6) $a_n = 3n + 5$
(7) $a_n = 3n - 7$
(8) $a_n = 3(5^n)$
(9) $a_n = -4n + 2$
(10) $a_n = 3^n$
(11) $a_n = 3^{n-3}$
(12) $a_n = 2a_{n-1}$ with $a_1 = 2$
(13) $a_n = 5a_{n-1}$ with $a_1 = 3$
(14) $*a_n = na_{n-1}$ with $a_1 = 1$
(15) $3, 6, 11, 18, 27, 38, \ldots$
(16) $.5, -.25, .125, -.0625, \ldots$

12.2. Series

WHAT IS...

A series is a sum of a bunch (finite or infinite many) of numbers. Said another way, a series is the sum of the terms of a sequence.

WAYS OF DESCRIBING

(1) (Using plain English) The sum of the squares of all integers, from 1 to 100.
(2) (By listing the first few terms of the sum)

$$1 + 2^2 + 3^2 + 4^2 + \cdots + 100^2$$

(3) (The professional Way, using summation sign). This upper case Greek letter sigma represents *sum*. Under the sigma we find the indexing variable, and its initial value. We sum each term of the summand sequence, until the index variable reaches the value on top of the sigma sign. The above series could be described as

$$\sum_{n=1}^{100} n^2$$

EXAMPLES

Calculate the following series

(1) $\sum_{k=3}^{7} k^2 - 3$

Solution:

$$\begin{aligned} \sum_{k=3}^{7} k^2 - 3 &= (3^2 - 3) + (4^2 - 3) + (5^2 - 3) + (6^2 - 3) + (7^2 - 3) && \text{Def of summation} \\ &= 6 + 13 + 22 + 33 + 46 && \text{By Inspection} \\ &= 120 && \text{B.I.} \end{aligned}$$

(2) $\sum_{t=1}^{3} (k - 1)(2k)$

Solution:

$$\begin{aligned}\sum_{t=1}^{3}(k-1)(2k) &= (1-1)(2\cdot 1)+(2-1)(2\cdot 2)+(3-1)(2\cdot 3) && \text{Def of Sum}\\ &= 0+4+12 && \text{B.I.}\\ &= 16 && \text{B.I.}\end{aligned}$$

(3) $\sum_{t=1}^{3}(\frac{t+1}{3}-\frac{1}{t})$

Solution:

$$\begin{aligned}\sum_{t=1}^{3}(\frac{t+1}{3}-\frac{1}{t}) &= (\frac{1+1}{3}-\frac{1}{1})+(\frac{2+1}{3}-\frac{1}{2})+(\frac{3+1}{3}-\frac{1}{3}) && \text{Def of Sum}\\ &= (\frac{2}{3}-\frac{1}{1})+(\frac{3}{3}-\frac{1}{2})+(\frac{4}{3}-\frac{1}{3}) && \text{B.I.}\\ &= \frac{7}{6} && \text{B.I.}\end{aligned}$$

(4) $\sum_{t=1}^{10}(3t+5)$

Solution:

$$\begin{aligned}\sum_{t=1}^{10}(3t+5) &= (3\cdot 1+5)+(3\cdot 2+5)+(3\cdot 3+5)+(3\cdot 4+5)+\\ &\quad (3\cdot 5+5)+\cdots+(3\cdot 9+5)+(3\cdot 10+5) && \text{Def of Sum}\\ &= 8+11+14+17+20+23+26+29+32+35 && \text{B.I.}\\ &= 215 && \text{B.I.}\end{aligned}$$

(5) $\sum_{t=1}^{20}(3t+5)$

Solution:

$$\begin{aligned}\sum_{t=1}^{10}(3t+5) &= (3\cdot 1+5)+(3\cdot 2+5)+(3\cdot 3+5)+(3\cdot 4+5)+ \\ &\quad (3\cdot 5+5)+\cdots+(3\cdot 19+5)+(3\cdot 20+5) && \text{Def of Sum} \\ &= 8+11+14+17+20+23+26+29+32+35\cdots+62+65 && \text{B.I.} \\ &= 635 && \text{PPNP}\end{aligned}$$

(6) $\sum_{t=11}^{20}(3t+5)$

Solution:

Note that the sum of these terms from the 11th term to the 20th term is the same as the adding the first 20 terms and then subtracting the first 10 terms. That is:

$$a_{11}+a_{12}+a_{13}+\cdots+a_{20} = (a_1+a_2+\cdots+a_{20}) - (a_1+a_2+\cdots+a_{10})$$

or said another way...

$$\sum_{t=11}^{20}(3t+5) = \sum_{t=1}^{20}(3t+5) - \sum_{t=1}^{10}(3t+5)$$

Thus...

$$\begin{aligned}\sum_{t=11}^{20}(3t+5) &= \sum_{t=1}^{20}(3t+5) - \sum_{t=1}^{10}(3t+5) && \text{Previous Remark} \\ &= 625-215 && \text{See previous examples} \\ &= 410 && \text{BI}\end{aligned}$$

EXAMPLES

Express sum using sigma notation

(1) 1+6+11+...+51

Solution:

Note this is an arithmetic sequence, terms increase by a constant of 5. Thus the sequence looks like $a_n = 5n + initialvalue$. Since we want to start at 1 we take initial value $= -4$ thus $a_n = 5n - 4$ is the closed form of the sequence, starting with $n = 1$ and ending with $n = 10$ (for $n = 10$ the term is 51, we have to stop adding at 51) thus the sum can be written as

$$\sum_{n=1}^{10} 5n - 4$$

(2) 3+5+7+...+81

Solution:

Note this is an arithmetic sequence, terms increase by a constant of 2. Thus the sequence looks like $a_n = 2n + initialvalue$. Since we want to start at 3 we take initial value = 1 thus $a_n = 2n + 1$ is the closed form of the sequence, starting with $n = 1$ and ending with $n = 40$ (the 40th term is 81, we must stop adding here)thus the sum can be written as

$$\sum_{n=1}^{40} 5n - 4$$

(3) 5+1+-3+-7+...+-47

Solution:

Note this is an arithmetic sequence, terms increase by a constant of -4. Thus the sequence looks like $a_n = -4n + initialvalue$. Since we want to start at 5 we take initial value = 9 thus $a_n = -4n + 9$ is the closed form of the sequence, starting with $n = 1$ and ending with $n = 14$ (the 14th term is -47, we must stop adding here)thus the sum can be written as

$$\sum_{n=1}^{17} -4n + 9$$

(4) 3+6+12+24+48+96

Solution:

Note this is an geometric sequence, terms increase by *multiplying* by 2. Thus the sequence looks like $a_n = (initialvalue) \cdot 2^n$. Since we want to start at 3 we take initial value = $\frac{3}{2}$ thus $a_n = \frac{3}{2} \cdot 2^n = 3 \cdot 2^{n-1}$ is the closed form of the sequence, starting with $n = 1$ and ending with $n = 6$ (the 6th term is 96, we must stop adding here)thus the sum can be written as

$$\sum_{n=1}^{16} 3 \cdot 2^{n-1}$$

(5) 2+2+2+2+2+2+2

Solution:

Note this is a constant sequence, all terms are 2. Thus the sequence looks like $a_n = 2$. Since there are 7 terms in the sum we start with $n = 1$ and end with $n = 7$. The sum can be written as

$$\sum_{n=1}^{7} 2$$

(6) 4+9+16+36+...+100

Solution:

Note this is a sequence of squares, the sequence is $a_n = n^2$. Starting with $n = 2$ and ending with $n = 10$. Thus the sum can be written as

$$\sum_{n=2}^{10} n^2$$

Excercises 12.2

(1) Calcualte the following sums
   (a) $\sum_{k=1}^{5} 3k - 2$
   (b) $\sum_{k=1}^{5} 5k - 3$
   (c) $\sum_{k=10}^{15} 3k - 2$
   (d) $\sum_{k=6}^{10} 5k + 3$
   (e) $*\sum_{k=1}^{100} 5k + 3$
   (f) $\sum_{k=2}^{6} 5k^2 + 3$
   (g) $\sum_{k=2}^{6} 2^k + 3$
   (h) $\sum_{k=1}^{100} 3$
   (i) $\sum_{k=49}^{55} 3$
   (j) $\sum_{k=3}^{7} 4k^2 - 3$
   (k) $\sum_{k=1}^{20} (k+1)^2 - k^2$
   (l) $\sum_{k=1}^{100} (\frac{1}{k+1} - \frac{1}{k})$
   (m) $\sum_{k=1}^{100} (\frac{k}{k+1} - \frac{k-1}{k})$
   (n) $\sum_{k=1}^{100} (k+1)^3 - k^3$
   (o) $*\sum_{k=1}^{100} k$
   (p) $*\sum_{k=1}^{100} k^2$
   (q) $*\sum_{k=1}^{100} k^3$

(2) First write using sigma notation, then compute the *finite* sums.
   (a) $1 + 2 + 3 + 4 + 5 \cdots + 10$
   (b) $5 + 11 + 17 + 23 + 29$
   (c) $7 + 14 + 28 + 56 + 112$
   (d) $100 + 90 + 80 + 20 + \cdots + -10$
   (e) $2 + 5 + 10 + 17 + 26 + 37 + 50 + 65$ (hint: almost squares)
   (f) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$
   (g) $1 + \frac{1}{1} + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2\cdot 1} + \frac{1}{4\cdot 3\cdot 2\cdot 1}$

12.3. Famous Series

Some famous Series Properties

(1) Constants factor through the summation sign

$$\sum ca_n = c\sum a_n$$

(2) Distributive Property

$$\sum (a_n + b_n) = \sum a_n + \sum b_n$$

Constant Series

Constant series are easy to recognize and easy so sum. The general principle is that

$$\sum_{n=1}^{t} c = tc$$

Telescoping Series

Telescoping sums are sums where all the middle terms cancel leaving the end terms alone. This feature makes them very easy to solve.

Telescoping Examples

(1) $\sum_{k=1}^{20}(k+1)^2 - k^2$

Solution:

$$\begin{aligned}\sum_{k=1}^{100}(k+1)^2 - k^2 &= 2^2 - 1^2 + 3^2 - 2^2 + 4^2 - 3^2 + 5^2 - 4^2 + \\ &\cdots + 100^2 - 99^2 + 101^2 - 100^2 && \text{Def of Sum} \\ &= 101^2 - 1^2 && \text{Cancel all middle terms} \\ &= 10200 && \text{BI}\end{aligned}$$

(2) $\sum_{k=1}^{100}(\frac{k}{k+1}-\frac{k-1}{k})$

Solution:

$$\begin{aligned}\sum_{k=1}^{100}(\frac{k}{k+1}-\frac{k-1}{k}) &= (\frac{1}{2}-\frac{0}{1})+(\frac{2}{3}-\frac{1}{2})+\cdots+(\frac{101}{102}-\frac{100}{101}) && \text{Def of Sum}\\ &= \frac{101}{102}-\frac{0}{1} && \text{Cancel all middle terms}\\ &= \frac{101}{102} && \text{BI}\end{aligned}$$

Young Gauss Series

So the story goes that as a young kid inside his mom's womb Karl Gauss figured out a very nice and quick way to add all the numbers from 1 to 100. Said another way, he quickly computed $\sum_{n=1}^{100} n$. We now show very elegant way to do this....

$$\begin{aligned}(n+1)^2-n^2 &= n^2+2n+1-n^2 && \text{FOIL}\\ (n+1)^2-n^2 &= 2n+1 && \text{BI}\\ \sum_{n=1}^{100}\{(n+1)^2-n^2\} &= \sum_{n=1}^{100}(2n+1) && \text{Cancellation Law of Sigma :)}\\ \sum_{n=1}^{100}\{(n+1)^2-n^2\} &= \sum_{n=1}^{100}2n+\sum_{n=1}^{100}1 && \text{DL}\\ \sum_{n=1}^{100}\{(n+1)^2-n^2\} &= 2\sum_{n=1}^{100}n+\sum_{n=1}^{100}1 && \text{Pull Constant Property}\\ 101^2-1^2 &= 2\sum_{n=1}^{100}n+\sum_{n=1}^{100}1 && \text{Telescoping Series on Left Hand Side}\\ 101^2-1^2 &= 2\sum_{n=1}^{100}n+100\cdot 1 && \text{Constant Series}\\ 10200 &= 2\sum_{n=1}^{100}n+100 && \text{BI}\\ 10100 &= 2\sum_{n=1}^{100}n && \text{CLA}\\ \frac{10100}{2} &= \sum_{n=1}^{100}n && \text{CLM}\\ 5050 &= \sum_{n=1}^{100}n && \text{BI}\end{aligned}$$

Notice by the 5th step you should be smiling! because by then its is clear that the left side is a telescoping sum, very easy to calculate, the right hand side has has a constant series, also very easy to compute. The only questionable part of the equation is the series $\sum_{n=1}^{100} n$, but that is perfect because we just solve for it by isolating it on one side of the equation and evaluating everything else. The there is nothing special about 100 above. If we replace 100 with a generic upper bound, k then we get the very famous Young-Gauss formula:

$$\sum_{n=1}^{k} n = \frac{k(k+1)}{2}$$

Sums of Squares

We seek a nice way of adding $\sum_{n=1}^{k} n^2$. We proceed to generalize the previous trick.

$$(n+1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 \qquad \text{Multiply}$$

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1 \qquad \text{Simplify}$$

$$\sum_{n=1}^{k}\{(n+1)^3 - n^3\} = \sum_{n=1}^{k}(3n^2 + 3n + 1) \qquad \text{Sum Both Sides}$$

$$\sum_{n=1}^{k}\{(n+1)^3 - n^3\} = \sum_{n=1}^{k} 3n^2 + \sum_{n=1}^{k} 3n + \sum_{n=1}^{k} 1 \qquad \text{Distributive Property}$$

$$\sum_{n=1}^{k}\{(n+1)^3 - n^3\} = 3\sum_{n=1}^{k} n^2 + 3\sum_{n=1}^{k} n + \sum_{n=1}^{k} 1 \qquad \text{Pull Constant Property}$$

$$(k+1)^3 - 1^3 = 3\sum_{n=1}^{k} n^2 + 3\sum_{n=1}^{k} n + \sum_{n=1}^{k} 1 \qquad \text{Telescoping Series}$$

$$(k+1)^3 - 1^3 = 3\sum_{n=1}^{k} n^2 + 3\frac{k(k+1)}{2} + k \cdot 1 \qquad \text{Y. G. Sum, and Constat Sum}$$

$$(k+1)^3 - 1^3 - 3\frac{k(k+1)}{2} - k = 3\sum_{n=1}^{k} n^2 \qquad \text{CLA}$$

$$k^3 + \frac{3}{2}k^2 + \frac{1}{2}k = 3\sum_{n=1}^{k} n^2 \qquad \text{Simplify}$$

$$\frac{1}{3}(k^3 + \frac{3}{2}k^2 + \frac{1}{2}k) = \sum_{n=1}^{k} n^2 \qquad \text{C.L.M}$$

$$\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k = \sum_{n=1}^{k} n^2 \qquad \text{Simplify}$$

$$\frac{1}{6}k\,(k+1)\,(2k+1) = \sum_{n=1}^{k} n^2 \qquad \text{Simplify}$$

EXAMPLES]

(1) $\sum_{n=1}^{50} 3n + 2$

Solution:

$$\begin{aligned}
\sum_{n=1}^{50} 3n+2 &= \sum_{n=1}^{50} 3n + \sum_{n=1}^{50} 2 && \text{D.P.}\\
&= 3\sum_{n=1}^{50} n + \sum_{n=1}^{50} 2 && \text{Pull Constant}\\
&= 3\sum_{n=1}^{50} n + 50\cdot 2 && \text{Constant Series}\\
&= 3\cdot\frac{50(51)}{2} + 50\cdot 2 && \text{Y.G. Series}\\
&= 3925 && \text{D.S.}
\end{aligned}$$

(2) $\sum_{n=1}^{50} -3n+5$

Solution:

$$\begin{aligned}
\sum_{n=1}^{50} -3n+5 &= \sum_{n=1}^{50} -3n + \sum_{n=1}^{50} 5 && \text{D.P.}\\
&= -3\sum_{n=1}^{50} n + \sum_{n=1}^{50} 5 && \text{Pull Constant}\\
&= -3\sum_{n=1}^{50} n + 50\cdot 5 && \text{Constant Series}\\
&= -3\cdot\frac{50(51)}{2} + 50\cdot 5 && \text{Y.G. Series}\\
&= -3575 && \text{D.S.}
\end{aligned}$$

(3) $\sum_{n=1}^{50} 3n^2-4n+2$

Solution:

$$\begin{aligned}
\sum_{n=1}^{50} 3n^2 - 4n + 2 &= \sum_{n=1}^{50} 3n^2 - \sum_{n=1}^{50} 4n + \sum_{n=1}^{50} 2 && \text{D.P.} \\
&= 3\sum_{n=1}^{50} n^2 - 4\sum_{n=1}^{50} n + \sum_{n=1}^{50} 2 && \text{Pull Constants} \\
&= 3 \cdot \frac{50}{6}(50+1)(2\cdot 50+1) - 4\cdot\frac{50\cdot 51}{2} + 50\cdot 2 && \text{Easy, famous Series} \\
&= 3 \cdot \frac{50}{6}(51)(101) - 4\cdot 25\cdot 51 + 50\cdot 2 && \text{Simplify} \\
&= 123775 && \text{D.S.}
\end{aligned}$$

EXCERCISES 12.3

Calculate

(1) $\sum_{n=1}^{50} -4n+1$
(2) $\sum_{n=1}^{100} 3n+5$
(3) $\sum_{n=1}^{100} 5n-7$
(4) $\sum_{n=1}^{50} 5n-7$
(5) $\sum_{n=51}^{100} 5n-7$
(6) $\sum_{n=1}^{50} -4n^2+1$
(7) $\sum_{n=1}^{100} 3n^2+5$
(8) Derive a formula for $\sum_{n=1}^{k} n^3$
(9) $\sum_{n=1}^{50} 5n^3-7$
(10) $\sum_{n=1}^{40}(2n^3-3n^2+5n-7)$
(11) $10+13+16+\cdots+103$

12.4. Geometric Series

Recall Famous Factoring

Recall the following famous factorizations of the famous geometric polynomials.

$$\begin{aligned} x^2 - 1 &= (x-1)(x+1) \\ x^3 - 1 &= (x-1)(x^2+x+1) \\ x^4 - 1 &= (x-1)(x^3+x^2+x+1) \\ x^5 - 1 &= (x-1)(x^4+x^3+x^2+x+1) \\ x^6 - 1 &= (x-1)(x^5+x^4+x^3+x^2+x+1) \\ x^{n+1} - 1 &= (x-1)(x^n+\cdots+x^4+x^3+x^2+x+1) \end{aligned}$$

The Brilliant Idea

The brilliant idea is to take any of the above equations and divide both sides by $x-1$. The result is the world famous formula for adding a geometric series.

$$\frac{x^{n+1}-1}{(x-1)} = x^n + \cdots + x^4 + x^3 + x^2 + x + 1$$

Examples

(1) Sum: $1 + 2 + 2^2 + 2^3$

Solution:

$$\begin{aligned} 1 + 2 + 2^2 + 2^3 &= \frac{2^4-1}{2-1} && \text{Geometric Series} \\ &= \frac{15}{1} && \text{B.I.} \\ &= 15 && \text{D.S.} \end{aligned}$$

(2) Sum: $\sum_{n=0}^{5} 3^n$

Solution:

$$\begin{aligned}\sum_{n=0}^{5} 3^n &= 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 && \text{Def sum}\\ &= \frac{3^6 - 1}{3 - 1} && \text{Geometric Series}\\ &= \frac{729 - 1}{2} && \text{B.I.}\\ &= 364 && \text{BI}\end{aligned}$$

(3) Sum $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64}$

Solution:

$$\begin{aligned}1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} &= \sum_{n=0}^{6} (\frac{-1}{2})^n && \text{Def of sum}\\ &= \frac{(\frac{-1}{2})^7 - 1}{(\frac{-1}{2}) - 1} && \text{G.S.}\\ &= \frac{43}{64} && \text{Simplify, Grunt Work}\end{aligned}$$

(4) *Sum $\sum_{n=0}^{\infty} (\frac{2}{3})^n$

Solution:

$$\begin{aligned}\sum_{n=0}^{\infty} (\frac{2}{3})^n &= \frac{(\frac{2}{3})^{\infty+1} - 1}{(\frac{2}{3}) - 1} && \text{G.S}\\ &= \frac{(\frac{2}{3})^{\infty} - 1}{-\frac{1}{3}} && \text{Simplify}\\ &= \frac{0 - 1}{-\frac{1}{3}} && \text{Observe } (\frac{2}{3})^{\infty} = 0\\ &= 3 && \text{Duck Soup!!}\end{aligned}$$

EXCERCISES 12.4

(1) $\sum_{n=0}^{7} 3^n$
(2) $\sum_{n=0}^{6} 2 \cdot 3^n$
(3) $\sum_{n=1}^{6} 2 \cdot 3^n$
(4) $\sum_{n=0}^{15} (\frac{-1}{3})^n$
(5) $\sum_{n=0}^{30} (\frac{1}{2})^n$
(6) $\sum_{n=0}^{15} 3 \cdot (\frac{1}{2})^n - 2 \cdot (\frac{2}{3})^n$
(7) *$\sum_{n=0}^{\infty} (\frac{1}{2})^n$
(8) *$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \frac{3}{100000} + \frac{3}{1000000} + \frac{3}{10000000} + \cdots$
(9) *Suppose a basketball is dropped from a 10ft height, and at each bounce the basketball returns to $\frac{4}{5}$ of the previous height. If left bouncing indefinitely, what distance will the ball travel before coming to a rest.
(10) *Use a geometric series to write $.12\overline{345}$ as an integer fraction.
(11) *At 12:00 the hour hand and the minute hand of a clock are lined up one exactly on top of the other. The minute hand takes off running faster than the hour hand. After 60 marks traveled by the minute hand the hour hand travels 5 marks. The minute hand tries to catch up and runs 5 marks, but by that time the hour hand has move ahead $\frac{5}{12}$ marks. The minute hand again tries to catch up by running $\frac{5}{12}$ marks, but the hour hand has moved again, this time $\frac{5}{144}$ marks. Will the minute hand ever catch up with the hour hand? At what is the exact time when this happens? (Hint: use geometric series for a very elegant solution)

Part 1

Solutions

CHAPTER 1

Basics

1.1. Preliminaries

1.2. Sets and Operations

1

$$5 \cdot 3 = 15 \qquad \text{TT}$$

2

$$5 + 2 = 7 \qquad \text{AT}$$

3 we don't know!!! (yet), 0 is not a natural number. We only learned how to add natural numbers

6 we don't know!!! (yet), -1 is not a natural number

7 we don't know!!! (yet), -1 is not a natural number

8 we don't know!!! (yet), -1 is not a natural number

1.3. Commutativity Axiom

1 explain in your own words

2 yes, COLA

3 yes, COLM

4 yes, if x is a number, COLM

5 yes, if $(x + y)$ is a number, COLM

6 yes, if s, i are numbers, by COLM

7 yes by COLA

8 yes by COLM

9 depends if log is a number and x is a number

10 only the order is changed, so yes, by COLM

11 only the order is changed, so yes, by COLA

12 only the order is changed, so yes assuming x is a number , by COLA

1.4. Associativity Axiom

2 yes by ALA
3 yes by ALM
4 yes by ALM
5 yes by ALM
6 yes by ALA
7 yes by ALM
8 yes by ALM
9 not necessarily, if 'log' is not a number
10 not necessarily, if 'sin' is not a number
11 patience, we will do this...

1.8. Exponents

1

$$5^4 = 5 \cdot 5 \cdot 5 \cdot 5 \qquad \text{Neg-Expo}$$

2

$$2^{-3} = \frac{1}{2^3} \qquad \text{Neg-Expo}$$

3

$$2x^2 = 2x \cdot x \qquad \text{Neg-Expo}$$

4

$$(2x)^2 = (2x)(2x) \qquad \text{Neg-Expo}$$

8

$$(x+1)^{-2} = \frac{1}{(x+1)^2} \qquad \text{Neg-Expo}$$

1.9. Paperplication

1.10. Review

1 True, TP
2 True, ALA
3 False
4 True, CLM
7 True, AInv
13 True, AInv

14 True, AInv

15 True, MInv

25 True, +Expo

26 True, +Expo

27 False

28 False

49 False as words, true as numbers

CHAPTER 2

Integers

2.1. Multiplying

1 False

2 True, OMT

3 True, NNT

4 False

20

$$\begin{aligned} (3+5)(2+5) &= 8 \cdot 7 && \text{TT} \\ &= 56 && \text{TT} \end{aligned}$$

21

$$\begin{aligned} (-1)^7 &= -1 \cdot -1 \cdot -1 \cdot -1 \cdot -1 \cdot -1 \cdot -1 && \text{+Expo} \\ &= 1 \cdot 1 \cdot 1 \cdot -1 && \text{NNT} \\ &= -1 && \text{NNT} \end{aligned}$$

23 1

24 -1

25 1

26 -1

36 -4

2.2. Adding

1o False

1p True, CoLA

1q True, CoLA

1r True, AId

2.3. Factoring

1 Yes

2 No

4 No

5 No

6 Yes

8 Yes
9 NO
12

$$\begin{aligned} 48 &= 4 \cdot 12 && \text{TT} \\ &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 && \text{TT} \\ &= 2^4 \cdot 3 && \text{+Expo} \end{aligned}$$

13 2^7
14 2^8

2.4. GCD

1 1
3 12
6 4
8 3
13 x^2
14 24
17 1
21 1
24 1997

2.5. LCM

1 100
7 222
14 48
22 $2^5 \cdot 3^7$
25 x^3yz^2

2.6. Integers Modulo 7

1 1
2 2
3 3
4 5
7 2
8 3 and 4 are both solutions
9 2
10 4

CHAPTER 3

Rationales

3.1. Multiplying $\mathbb{Q}$

5 $\frac{14}{42}$

6 $\frac{8blah}{98}$

9 $\frac{20}{28}$

17 1

20 $\frac{18}{5}$

3.2. Simplifying $\mathbb{Q}$

1 $\frac{6}{-7}$

2 $\frac{27}{-6}$

5 $\frac{2x^2}{5}$

8 $\frac{2}{5x^2}$

10 $\frac{-3}{5}$

12 $\frac{3x^5}{5}$

16 $\frac{8}{9}$

17 $\frac{r}{t}$

18 can't simplify

19 $\frac{8}{7}$

3.3. Dividing $\mathbb{Q}$

5 $\frac{-20}{-30}$ or $\frac{2}{3}$ (show work to simplify, if you simplify)

6 1 by JOT

7 $\frac{14}{15}$

9

$$
\begin{aligned}
\frac{-7}{3} \div \frac{\frac{2}{3}}{\frac{2}{7}} &= \frac{-7}{3} \cdot \left(\frac{\frac{2}{3}}{\frac{2}{7}}\right)^{-1} && \text{Def of } \div \\
&= \frac{-7}{3} \cdot \frac{\frac{2}{7}}{\frac{2}{3}} && \text{MInv (or Neg Expo)} \\
&= \frac{\frac{-7}{3}}{1} \cdot \frac{\frac{2}{7}}{\frac{2}{3}} && \text{OUT} \\
&= \frac{\frac{-7}{3} \cdot \frac{2}{7}}{1 \cdot \frac{2}{3}} && \text{MAT} \\
&= \frac{\frac{-14}{21}}{\frac{2}{3}} && \text{MAT, TT, NPT} \\
&= \frac{-14}{21} \div \frac{2}{3} && \text{def } \div, \frac{a}{b} \\
&= \frac{-14}{21} \cdot \left(\frac{2}{3}\right)^{-1} && \text{def } \div \\
&= \frac{-14}{21} \cdot \frac{3}{2} && \text{MInv} \\
&= \frac{-14 \cdot 3}{21 \cdot 2} && \text{MAT} \\
&= \frac{-42}{42} && \text{NPT, TT} \\
&= -\frac{42}{42} && \text{NWW} \\
&= -1 && \text{JOT}
\end{aligned}
$$

3.4. Adding $\mathbb{Q}$

1 $\frac{49}{-5}$
2 $\frac{69}{9}$
4 $\frac{8}{5}$ or $\frac{-16}{-10}$
7 $\frac{-9}{18}$ or $\frac{-1}{2}$
11 $\frac{28}{6}$ or $\frac{14}{3}$
12 $\frac{10}{2x}$
13 $\frac{17}{41}$?

CHAPTER 4

Polynomials

4.1. one

1 $\frac{1}{x^6y^5}$

2 $\frac{y^2}{x^2}$

3 $\frac{1}{x^2}$

4 $\frac{1}{xy}$

5 x^3y^2

6 $\frac{1}{x^5y^{10}}$

7 $2x^7y^{10}$

8 x^4y^3

10 xy^4

15 xy^2

4.2. two

4.3. three

1 $5x^3 +^- x + 3$

2 $8x^2 + (\pi +^- 4)x$

3 $^-2x^2 +^- 7x + 4$

6 $4x^3$

9 Do not *add* degree-2 terms with degree-1 terms...

4.4. four

1

3 $7x^4 + 5x^5 - x^3 + 5x^2$

8 $1 + 2x + x^2$

9 $1 - x^2$

11 $a^2 + 2ab + b^2$

12 $a^2 - 2ab + b^2$

13 $a^3 + 3a^2b + 3ab^2 + b^3$

14 $a^3 - 3a^2b + 3ab^2 - b^3$

20 $x^3 - 1$

21 $x^4 - 1$

22 $x^5 - 1$

4.5. Divide

4

$$
\begin{array}{r}
25x^2 + \frac{121}{3}x + 17 \\
3x^2 \overline{\big)\ 75x^4 + 121x^3 + 51x^2} \\
\underline{-\,75x^4} \qquad\qquad\qquad\quad \\
121x^3 \qquad\quad\ \\
\underline{-\,121x^3} \qquad\quad\ \\
51x^2 \\
\underline{-\,51x^2} \\
0
\end{array}
$$

5

$$
\begin{array}{r}
25x^3 + \frac{121}{3}x^2 + 17x + 2 \\
3x \overline{\big)\ 75x^4 + 121x^3 + 51x^2 + 6x + 1} \\
\underline{-\,75x^4} \qquad\qquad\qquad\qquad\qquad\quad \\
121x^3 \qquad\qquad\qquad\quad \\
\underline{-\,121x^3} \qquad\qquad\qquad\quad \\
51x^2 \qquad\qquad \\
\underline{-\,51x^2} \qquad\qquad \\
6x \quad\ \ \\
\underline{-\,6x} \quad\ \
\end{array}
$$

6

$$
\begin{array}{r}
25x^4 + \frac{121}{3}x^3 + 17x^2 + 2x + \frac{5}{3} \\
3 \overline{\big)\ 75x^4 + 121x^3 + 51x^2 + 6x + 5} \\
\underline{-\,75x^4} \qquad\qquad\qquad\qquad\qquad\quad \\
121x^3 \qquad\qquad\qquad\quad \\
\underline{-\,121x^3} \qquad\qquad\qquad\quad \\
51x^2 \qquad\qquad \\
\underline{-\,51x^2} \qquad\qquad \\
6x \qquad \\
\underline{-\,6x} \qquad \\
5 \\
\underline{-\,5} \\
0
\end{array}
$$

7

$$
\begin{array}{rrrrrr}
 & & 25x^3 + & \frac{121}{3}x^2 + & 17x + & 2 \\
\cline{2-5}
3x) & 75x^4 + & 121x^3 & +51x^2 & +6x & +1 \\
 & -75x^4 & & & & \\
\cline{2-2}
 & & 121x^3 & & & \\
 & & -121x^3 & & & \\
\cline{3-3}
 & & & 51x^2 & & \\
 & & & -51x^2 & & \\
\cline{4-4}
 & & & & 6x & \\
 & & & & -6x & \\
\cline{5-5}
\end{array}
$$

8

$$
\begin{array}{rrrrrr}
 & & \frac{5}{3}x^3 + & 4x^2 + & \frac{5}{3}x + & \frac{1}{3} \\
\cline{2-5}
3x) & 5x^4 + & 12x^3 + & 5x^2 & +x & +1 \\
 & -5x^4 & & & & \\
\cline{2-2}
 & & 12x^3 & & & \\
 & & -12x^3 & & & \\
\cline{3-3}
 & & & 5x^2 & & \\
 & & & -5x^2 & & \\
\cline{4-4}
 & & & & x & \\
 & & & & -x & \\
\cline{5-5}
\end{array}
$$

9

$$
\begin{array}{rrrrrr}
 & 25x^4 & + \frac{121}{3}x^3 & +17x^2 & +2x & +5 \\
\cline{2-6}
3) & 75x^4 & +121x^3 & +51x^2 & +6x & +15 \\
 & -75x^4 & & & & \\
\cline{2-2}
 & & 121x^3 & & & \\
 & & -121x^3 & & & \\
\cline{3-3}
 & & & 51x^2 & & \\
 & & & -51x^2 & & \\
\cline{4-4}
 & & & & 6x & \\
 & & & & -6x & \\
\cline{5-5}
 & & & & & 15 \\
 & & & & & -15 \\
\cline{6-6}
 & & & & & 0
\end{array}
$$

4.6. Divide

1

$$
\begin{array}{rrrrr}
 & & 6x^2 & +23x & +27 \\ \cline{2-5}
x-2) & 6x^3 & +11x^2 & -19x & +10 \\
 & -6x^3 & +12x^2 & & \\ \cline{2-3}
 & & 23x^2 & -19x & \\
 & & -23x^2 & +46x & \\ \cline{3-4}
 & & & 27x & +10 \\
 & & & -27x & +54 \\ \cline{4-5}
 & & & & 64
\end{array}
$$

2

$$
\begin{array}{rrrrr}
 & & 6x^2 & -7x & +2 \\ \cline{2-5}
x+3) & 6x^3 & +11x^2 & -19x & +10 \\
 & -6x^3 & -18x^2 & & \\ \cline{2-3}
 & & -7x^2 & -19x & \\
 & & 7x^2 & +21x & \\ \cline{3-4}
 & & & 2x & +10 \\
 & & & -2x & -6 \\ \cline{4-5}
 & & & & 4
\end{array}
$$

3

$$
\begin{array}{rrrrr}
 & & 2x^2 & +3x & -3 \\ \cline{2-5}
3x+1) & 6x^3 & +11x^2 & -6x & +1 \\
 & -6x^3 & -2x^2 & & \\ \cline{2-3}
 & & 9x^2 & -6x & \\
 & & -9x^2 & -3x & \\ \cline{3-4}
 & & & -9x & +1 \\
 & & & 9x & +3 \\ \cline{4-5}
 & & & & 4
\end{array}
$$

4

$$
\begin{array}{rrrrr}
 & & 2x^2 & +5x & -3 \\ \cline{2-5}
3x-2) & 6x^3 & +11x^2 & -19x & +10 \\
 & -6x^3 & +4x^2 & & \\ \cline{2-3}
 & & 15x^2 & -19x & \\
 & & -15x^2 & +10x & \\ \cline{3-4}
 & & & -9x & +10 \\
 & & & 9x & -6 \\ \cline{4-5}
 & & & & 4
\end{array}
$$

5

$$
\begin{array}{r|r}
 & 3x^2 + 10x + \frac{11}{2} \\
\hline
2x - 3) & 6x^3 + 11x^2 - 19x + 10 \\
 & -\, 6x^3 \;\; + 9x^2 \qquad\qquad\quad \\
\cline{2-2}
 & 20x^2 - 19x \qquad \\
 & -\, 20x^2 + 30x \qquad \\
\cline{2-2}
 & 11x + 10 \\
 & -\, 11x + \frac{33}{2} \\
\cline{2-2}
 & \frac{53}{2}
\end{array}
$$

6

$$
\begin{array}{r|r}
 & x^2 + x + 1 \\
\hline
x - 1) & x^3 \qquad\qquad - 1 \\
 & -\, x^3 + x^2 \qquad\quad \\
\cline{2-2}
 & x^2 \qquad\quad \\
 & -\, x^2 + x \qquad \\
\cline{2-2}
 & x - 1 \\
 & -\, x + 1 \\
\cline{2-2}
 & 0
\end{array}
$$

7

$$
\begin{array}{r|r}
 & x^3 + x^2 + x + 1 \\
\hline
x - 1) & x^4 \qquad\qquad\qquad - 1 \\
 & -\, x^4 + x^3 \qquad\qquad\quad \\
\cline{2-2}
 & x^3 \qquad\qquad\quad \\
 & -\, x^3 + x^2 \qquad\quad \\
\cline{2-2}
 & x^2 \qquad\quad \\
 & -\, x^2 + x \qquad \\
\cline{2-2}
 & x - 1 \\
 & -\, x + 1 \\
\cline{2-2}
 & 0
\end{array}
$$

8

$$
\begin{array}{r|r}
 & x^4 + x^3 + x^2 + x + 1 \\
\hline
x - 1) & x^5 \qquad\qquad\qquad\qquad - 1 \\
 & -\, x^5 + x^4 \qquad\qquad\qquad\quad \\
\cline{2-2}
 & x^4 \qquad\qquad\qquad\quad \\
 & -\, x^4 + x^3 \qquad\qquad\quad \\
\cline{2-2}
 & x^3 \qquad\qquad\quad \\
 & -\, x^3 + x^2 \qquad\quad \\
\cline{2-2}
 & x^2 \qquad\quad \\
 & -\, x^2 + x \qquad \\
\cline{2-2}
 & x - 1 \\
 & -\, x + 1 \\
\cline{2-2}
 & 0
\end{array}
$$

4.7. Factor DL

1 $10(x^3 + 2x^5 - 4x^2 + 10)$

2 $4(3x^3+6x^2+4)$
3 $x(10+14x^2-35x)$
4 $Z(2+x)$
5 $R(2+x)$
6 $\clubsuit(2+x)$
7 $stuff(2+x)$
8 $(x+1)(2+x)$
9 $(2x+1)(2+x)$
10 $(2x+1)\left(2+x^2+^- 3x^5\right)$
13 $(x+3)(2+x+7y)$
15 $x^5(3+2x^2)$
18 $(x^2+3x+5)(1+x^3)$
19 $(x^2+3x+5)(5+3x+x^2)$

4.8. Factor grouping

1 $4(2+x^3)(1+2x^2)$
2 (you should factor completely, as much as you can...)

$$\begin{aligned}
6-6y+8y^3-8y^4 &= 6+^- 6y+8y^3+^- 8y^4 && \text{def a-b}\\
&= 6(1+^- 1y)+8y^3(1+^- 1y && \text{DL}\\
&= (6+8y^3)(1+^- 1y && \text{DL}\\
&= (2\cdot 3+2\cdot 4y^3)(1+^- 1y && \text{TT}\\
&= 2(3+4y^3)(1+^- 1y && \text{TT}
\end{aligned}$$

3

$$\begin{aligned}
8-2x^2+12x^3-3x^5 &= 8+^- 2x^2+12x^3+^- 3x^5 && \text{def a-b}\\
&= 2(4+^- x^2)+3x^3(4+^- x^2) && \text{DL}\\
&= (2+3x^3)(4+^- x^2) && \text{DL}
\end{aligned}$$

(next sections we will learn how to factor more...)

4.9. More Factoring

1 $\big(x-2\big)\,\big(x+10\big)$
2 $\big(x+3\big)\,\big(x+10\big)$
3 $\big(x-1\big)\,\big(x+10\big)$
4 $\big(x-1\big)\,\big(x+10\big)$
5 $\big(x-7\big)\,\big(x+10\big)$
6 $\big(x+3\big)\,\big(x+10\big)$
7 $\big(x-2\big)\,\big(x+2\big)$

8 $(x-3)(x+10)$

9 $(x-1)(x-7)$

10 $(x+3)(x+5)$

11 $(x-1)(x+10)$

12 $(x+5)(x+10)$

13 $(x+2)(x+5)$

14 $(x-2)(x+4)$

15 $(x+2)(x-7)$

16 $(x+2)(x-7)$

17 $(x-7)(x+10)$

18 $(x-2)(x+5)$

19 $(x-3)(x-7)$

20 $(x-2)(x-3)$

21 $4x(x-1)(x+4)$

22 $-2x(x-3)(x-7)$

23 $4x(x-3)^2$

24 $12(x+1)(x-\frac{1}{2})$

25 $4(x+\frac{1}{2})(x-2)$

26 $6(x-\frac{1}{2})(x+\frac{4}{3})$

27 $3(x+1)(x+2)$

28 $3x^2(x-4)$

29 $2x(x+10)^2$

30 $-2x(x-3)(x+5)$

31 $2x(x+3)(x+4)$

32 $-2x(x-2)(x-7)$

33 $4x(x-3)(x+5)$

34 $-2x(x-7)^2$

35 $2x(x+5)(x+10)$

36 $4x(x-3)(x+4)$

37 $-1x(x-2)(x+10)$

38 $2x(x+3)(x+10)$

39 $2x(x+2)(x+3)$

40 $-2(x-1)(x-4)$

41 $-1(x+2)(x+3)$

42 $-4(x+1)(x-3)$

43 $2(x-2)(x-\frac{3}{2})$

44 $16\left(x+1\right)\left(x+\frac{1}{2}\right)$
45 $12\left(x+1\right)\left(x+\frac{3}{4}\right)$
46 $2\left(x+2\right)^2$
47 $-4\left(x-1\right)\left(x+2\right)$
48 $-1\left(x+1\right)\left(x-2\right)$
49 $4\left(x-1\right)^2$
50 $\left(8x^6-8x^3+2\right)$
51 $\left(-6x^6+5x^3+6\right)$
52 $\left(-2x^5-2x^4-2x^3-6x^2-6x-6\right)\left(x-1\right)$
53 $\left(6x^5-6x^4+6x^3+8x^2-8x+8\right)\left(x+1\right)$
54 $\left(-2x^5-2x^4-2x^3-4x^2-4x-4\right)\left(x-1\right)$
55 $-8\left(x+\frac{1}{2}\right)\left(x-\frac{3}{2}\right)$
56 $6\left(x-\frac{1}{3}\right)\left(x+2\right)$
57 $16\left(x+1\right)\left(x-\frac{1}{4}\right)$
58 $4\left(x+3\right)\left(x+\frac{3}{4}\right)$
59 $9\left(x+1\right)\left(x+\frac{4}{3}\right)$
60 $-3\left(x+1\right)\left(x+\frac{4}{3}\right)$
61 $4\left(x+1\right)\left(x+3\right)$

4.10. Famous Factoring

1 $(x+y)^2$
2 $(x+y)(x^2-xy+y^2)$
3 $(2x+3)(4x^2-6x+9)$
4 $(2x+5)(4x^2-10x+25)$
5 $(2x-5)(4x^2+10x+25)$
6 $(t+s)^2$
7 $(\heartsuit+\$)^3$
8 $(x+3)^3$
9 $(3x-2)(9x^2+6x+4)$
11 $(x+5)^2$
12 $(x+3)^2$
13 $(x+7)^2$
14 $(x+10)^2$
15 $\left(x+\frac{3}{2}\right)^2$
16 $\left(x+\frac{-7}{2}\right)^2$
17 $\left(x+\frac{1}{3}\right)^2$
18 $\left(x+\frac{-3}{10}\right)^2$

19 $(4x - 5)(4x + 5)$
20 Not Famous, yet...
21 $(3x - 1)(3x + 1)$
22 $(3x - 2)(3x + 2)$
23 $(3x + 2)^2$

4.11. Factoring All

4.12. Chapter Review